Application of the multivariate Loewner Framework to the modelling of acoustic propagation in rigid-frame porous materials

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Abstract

The propagation of acoustic waves in rigid-frame porous materials is commonly described using the Equivalent Fluid (EF) model, which depends on multiple parameters such as porosity and permeability. These porous properties are usually not accurately known, thus affecting the predicted acoustic response. In this work, reduced-order modelling is employed to approximate with controlled accuracy the parametrized EF model with a rational multivariate representation, using an extension of the Loewner Framework (LF), namely the multivariate LF (mLF) [1]. The latter is a data-driven approach allowing for constructing rational n-variable models by means of interpolation. It is adapted for constructing of models that depend on parameters, describing either uncertainties or parametric dependences. The main result of [1] concerns the variable decoupling achieved in the LF, providing a solution to the Kolmogorov Superposition Theorem restricted to rational functions. As a byproduct, it tames the curse of dimensionality in storage, computational complexity and accuracy, allowing dealing with a high number of parameters.

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1 Multivariate Loewner Framework

The mLF rationale consists in re-arranging the data, computing the barycentric coefficients via the Loewner matrix, and constructing the rational interpolant [1].

Tensor grid data. The data-set (tensor grid) is the primary ingredient. It is obtained by evaluating the n-variable function $\mathbf{H}(x_1, \cdots, x_n)$, either through simulations or experiments along the discretized grid $\mathbf{x}_1, \cdots, \mathbf{x}_n$ [1]. In the context of the LF, grid points $\mathbf{x}_l$ are split into columns $\lambda_l(j_l)$ and rows $\mu_l(i_l)$ interpolation, or support points[2]. Evaluating the multivariate function $\mathbf{H}$ along with the combinations of the support points $\lambda_l(j_l), \mu_l(i_l) \in \mathbb{C}$ thus forms an n-dimensional tensor, denoted $\mathcal{T}_n^{\otimes}$ [3]. Following the Loewner philosophy detailed in [1], we define $P_c^{(n)}$, the column data, and $P_r^{(n)}$, the row data, being two subsets of $\mathcal{T}_n^{\otimes}$, leading to $\mathbf{w}_{j_1,j_2,\cdots,j_n}$ and $\mathbf{v}_{i_1,i_2,\cdots,i_n}$:

$$\begin{aligned} P_c^{(n)} &: \{(\lambda_1(j_1), \lambda_2(j_2), \cdots, \lambda_n(j_n); \mathbf{w}_{j_1,j_2,\cdots,j_n})\} \\ P_r^{(n)} &: \{(\mu_1(i_1), \mu_2(i_2), \cdots, \mu_n(i_n); \mathbf{v}_{i_1,i_2,\cdots,i_n})\} . \end{aligned}$$

Multivariate Loewner matrix. The tensor data now serves for constructing the n-D Loewner matrix $\mathbb{L}_n$, which may be viewed as an operator mapping the interpolation points and $\mathcal{T}_n^{\otimes}$ onto a $Q \times K$ matrix, with $Q = q_1 q_2 \dots q_n$ (rows) and $K = k_1 k_2 \dots k_n$ (columns), where each entry of the $\mathbb{L}_n$ matrix is given by

$$\ell_{j_1,j_2,\cdots,j_n}^{i_1,i_2,\cdots,i_n} = \frac{\mathbf{v}_{i_1,i_2,\cdots,i_n} - \mathbf{w}_{j_1,j_2,\cdots,j_n}}{(\mu_1(i_1) - \lambda_1(j_1)) \cdots (\mu_n(i_n) - \lambda_n(j_n))}.$$

Lagrangian (barycentric) rational model. By considering an appropriate number of column interpolation points k_l ($l = 1, \cdots, n$), one can compute $\mathbb{L}_n \mathbf{c}_n = 0$, the right null space of $\mathbb{L}_n$, which contains the so-called barycentric coefficients, $\mathbf{c}_n^\top = [c_{1,\cdots,1}, \cdots, c_{1,\cdots,k_n} | \cdots, c_{k_1,\cdots,k_n}] \in \mathbb{C}^K$. Then, the multivariate Lagrangian (or barycentric) form $\mathbf{G}_{\text{lag}}(x_1, \cdots, x_n) =$

$$\frac{\sum_{j_1=1}^{k_1} \cdots \sum_{j_n=1}^{k_n} \frac{c_{j_1,\cdots,j_n} \mathbf{w}_{j_1,\cdots,j_n}}{(x_1 - \lambda_1(j_1)) \cdots (x_n - \lambda_n(j_n))}}{\sum_{j_1=1}^{k_1} \cdots \sum_{j_n=1}^{k_n} \frac{c_{j_1,\cdots,j_n}}{(x_1 - \lambda_1(j_1)) \cdots (x_n - \lambda_n(j_n))}},$$

[1] $\mathbf{x}_l = [x_l(1), \cdots, x_l(N_l)]$ and $x_l(i)$ is the l-th variable evaluated at the i-th element (with $l = 1, \cdots, n$).

[2] We also denote $j_l = 1, \cdots, k_l$ and $i_l = 1, \cdots, q_l$ with $l = 1, \cdots, n$ and $N_l = k_l + q_l$.

[3] We define $\mathbf{x}_1 = [\lambda_1(j_1), \mu_1(i_1)]$, $\mathbf{x}_2 = [\lambda_2(j_2), \mu_2(i_2)]$, $\dots$, $\mathbf{x}_n = [\lambda_n(j_n), \mu_n(i_n)]$.

interpolates the n-D data tensor and eventually reveals the original underlying function $\mathbf{H}$ (if rational). Reducing either k_l, or directly K (the null space entries), reduces the complexity of $\mathbf{G}_{\text{lag}}$ and leads to tensor approximation.

Decoupling of variables. The principal bottleneck of the above Loewner-driven model construction is the need for constructing a $Q \times K$ matrix and computing a K dimensional null space. In [1, Theorem 5.8], the n-D Loewner null space $\mathbf{c}_n$ is expressed as a linear combination of a collection of 1-D Loewner matrix null spaces. A direct consequence is [1, Theorem 5.9] which states how to decouple the evaluation of the barycentric coefficients, reducing complexity and memory cost [[1, Theorems 10 & 13], allowing the framework to be applied to large multi-dimensional tensors.

2 Porous media modelling

In this study, the mLF is applied to the EF model of rigid-frame porous media, that describes the homogenised behaviour of the acoustic pressure and velocity fields, governed by the coupled PDEs given in the Laplace domain by:

$$\left\{\begin{array}{rcl} \rho_0\, \alpha(s,\mathbf{x}) \cdot s\hat{\mathbf{u}} & = & -\nabla \hat{p}, \\ \chi_0\, \beta(s,\mathbf{x}) s\hat{p} & = & -\nabla \cdot \hat{\mathbf{u}}, \end{array}\right.$$

where ρ_0 , $\chi_0 \in \mathbb{R}^+$ are the ambient fluid density and compressibility, p the spatial average of the acoustic pressure in the porous domain and $\mathbf{u}$ the spatial average of the particle velocity in the pore, normalised by the total volume. The symbol $\hat{\cdot}$ denotes the Laplace transform and $s = \mathrm{j}\omega$ the associated complex variable.

The functions $\alpha(s,\mathbf{x})$ and $\beta(s,\mathbf{x})$ are irrational multivariate functions of s and of the parameter vector $\mathbf{x} = [\phi, \overline{r}, \sigma_r]$, respectively pore mean size, pore size standard deviation and porosity, all independent positive real numbers. They are defined as [2]:

$$\left\{\begin{array}{l} \alpha(s,\mathbf{x}) = \frac{e^{4S_2}}{\phi}\left[1+\frac{a_1}{\mathrm{j}\omega\overline{r}^2}e^{6S_2}+\frac{a_1 e^{6S_2}\sqrt{1+\frac{\mathrm{j}\omega\overline{r}^2}{2a_1}e^{-7S_2}}-1}{\mathrm{j}\omega\overline{r}^2}\right], \\ \beta(s,\mathbf{x}) = \phi\gamma-\phi(\gamma-1)\left[1+\frac{a_2 e^{-2S_2}}{\mathrm{j}\omega\overline{r}^2}+\frac{a_2 e^{-2S_2}\sqrt{1+\frac{\mathrm{j}\omega\overline{r}^2 e^{3S_2}}{a_3}}-1}{\mathrm{j}\omega\overline{r}^2}\right]^{-1} \end{array}\right.$$

Noting $S_2 = (\sigma_r \log_{10} 2)^2$ and γ, $a_{1,2,3}$ constant positive values associated to the ambient fluid.

The acoustic behaviour can then be described by the material impedance Z or absorption coefficient A, with h the material thickness and Z_0, $c_{1,2}$ constant positive values associated to the ambient fluid:

$$Z=-\mathrm{j}\sqrt{\frac{c_1\alpha}{\beta}}\cot\left(\omega h\sqrt{c_2\alpha\beta}\right) \text{ and } A=1-\left|\frac{Z-Z_0}{Z+Z_0}\right|^2 .$$

3 Application

The mLF is then applied on the porous media EF model constitutive functions $\alpha(s,\mathbf{x})$ and $\beta(s,\mathbf{x})$. This method identifies the optimal approximation order for each parameter, following a user-defined threshold ϵ depending on the required approximation fidelity (Fig. 1).

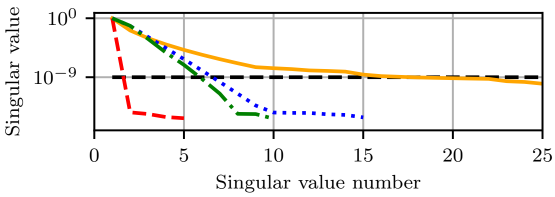

Figure 1: *Singular values for ω (—), ϕ (- -), $\overline{r}$ (···) and σ_r (-·-·). Threshold ϵ (- -).*

From the approximated functions, we can then get the absorption coefficient for different parameter configurations to validate the approximation (Fig. 2).

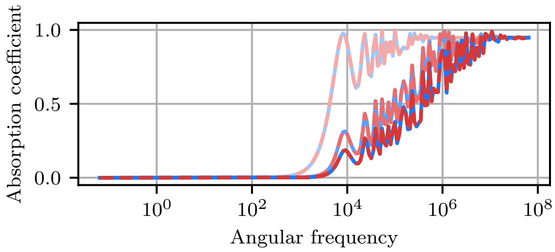

Figure 2: *Absorption coefficient from frequency model (—) and from its approximation (- -) for three different pore mean sizes (darker shade for bigger pores). Mean absolute error of order* 10^{-5}.

By partial fraction decomposition with fixed parameters $\mathbf{x} = [x_1 = \phi, x_2 = \overline{r}, x_3 = \sigma_r]$, the multivariate Lagrangian form $\mathbf{G}_{\text{lag}}(s, x_1, \cdots, x_n)$ can then be expressed in the Laplace domain, $G(s) = \sum_{k=1}^{k_1} \frac{r_k(\mathbf{x})}{s+\xi_k(\mathbf{x})}$, or in the time-domain, $g(t) = \sum_{k=1}^{k_1} r_k(\mathbf{x})\, e^{-\xi_k(\mathbf{x})t} 1_{t>0}$. This allows for an $\mathbf{x}$-dependent time-domain expression of the acoustic behaviour of porous media.

References

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