The Loewner framework, the Kolmogorov superposition theorem, and the curse of dimensionality

... from tensor to multivariate rational approximation and more

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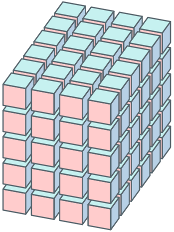
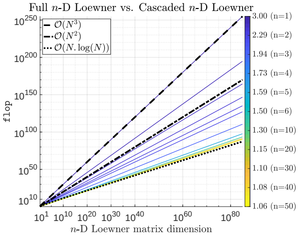
https://arxiv.org/abs/2405.00495 (to appear in SIAM Review - Research Spotlight)

https://github.com/cpoussot/mLF (GitHub code)

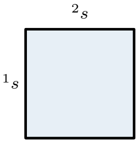




Forewords

Starting (motivating) examples - toy case

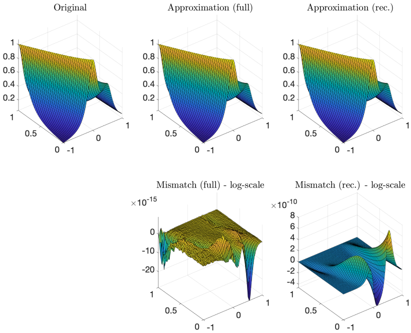
$$\mathbf{H}({}^1s, {}^2s) = \mathbf{H}(s, p) = \frac{1}{1 + 25(s+p)^2} + \frac{0.5}{1 + 25(s-0.5)^2} + \frac{0.1}{p + 25}$$



$$\begin{array}{ccc} {}^1s & \times & {}^2s \\ [-1, 1] & \times & [0, 1] \end{array}$$

tab$_2 \in \mathbb{R}^{21\times 21}$

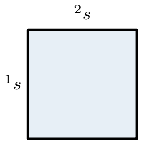
≈`3.45 Ko ('real')`



Forewords

Starting (motivating) examples - ReLU

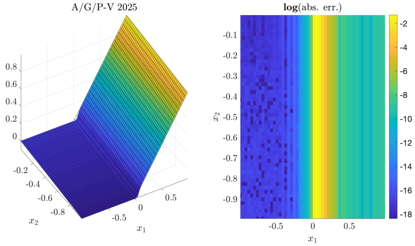
$$\mathbf{H}({}^1s, {}^2s) = \mathbf{ReLU}({}^1s) + \frac{1}{100}{}^2s$$



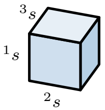
$$\begin{array}{ccc} {}^1s & \times & {}^2s \\ [-1,1] & \times & [0,1] \end{array}$$

$\textsf{tab}_2 \in \mathbb{R}^{20\times 20}$

≈`3.45 Ko ('real')`



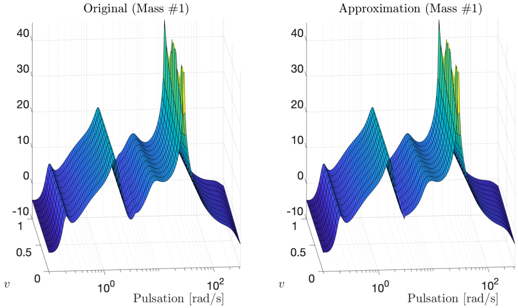
$$\mathbf{\Sigma}({}^1s, {}^2s, {}^3s) = \mathbf{\Sigma}(s, m, v) : s^2 M(m)x(s) + sB(m)x(s) + K(m)x(s) - G(s, v) = u(s),\ \mathbf{y}(s) = C\mathbf{x}(s)$$



$$\begin{array}{ccccc} {}^1s & \times & {}^2s & \times & {}^3s \\ \imath[10, 35] & \times & [\underline{m}, \overline{m}] & \times & [\underline{v}, \overline{v}] \end{array}$$

$\textbf{tab}_3 \in \mathbb{C}^{300\times10\times10}$

≈`468.75 Ko ('complex')`

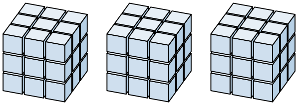


A. dos Reis de Souza et al., *"Aircraft flutter suppression: from a parametric model to robust control"*, ECC, 2023.

Forewords

Starting (motivating) examples - Borehole function

$$\mathbf{H}({}^1s,\cdots,{}^8s)=\mathbf{H}(r_w,r,T_u,H_u,T_l,H_l,L,K_w)=\frac{2\pi T_u(H_u-H_l)}{\ln\left(\frac{r}{r_w}\right)\left(1+\frac{2LT_u}{\ln(r/r_w)r_w^2K_w}\right)+\frac{T_u}{T_l}}$$

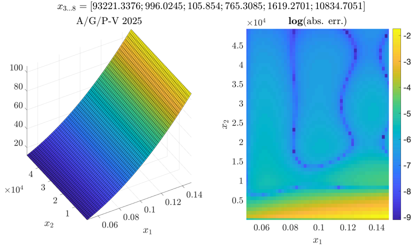


$$\begin{array}{ccccc} {}^1s & \times & \cdots & \times & {}^8s \\ [\underline{r_w},\overline{r_w}] & \times & \cdots & \times & [\underline{K_w},\overline{K_w}] \end{array}$$

$\textbf{tab}_8 \in \mathbb{C}^{8\times 8\times\cdots\times 8}$

≈130 Mo ('real')

#	Alg.	Parameters	Dim.	CPU [s]	RMSE	min err.	max err.
30	A1	1e-09,1	$1.02e+04$	19.3	0.00455	2e-09	0.061
	A2	1e-15,2	1.02e+04	39.1	0.00456	2.93e-09	0.0611



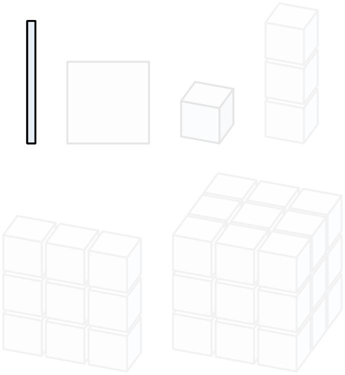
S. Surjanovic, *"Borehole function"*, `https://www.sfu.ca/~ssurjano/borehole.html`.

Column / Row data

$$\left.{}^{1}\lambda_{j_1}, {}^{1}\mu_{i_1}\right\} \xrightarrow{\mathbf{H}({}^{1}s)} \left\{ \mathbf{w}_{j_1}, \mathbf{v}_{i_1}\right.$$

${}^{1}s$	
${}^{1}\lambda_{1,\cdots,k_1}$	$\mathbf{W}_{k_1}$
${}^{1}\mu_{1,\cdots,q_1}$	$\mathbf{V}_{q_1}$

Tensors (1-D) tab_1



Forewords

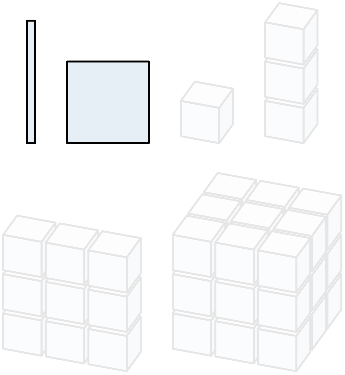
Data (and tensors)

Column / Row data

$$\left.\begin{array}{l} {}^1\lambda_{j_1}, {}^1\mu_{i_1} \\ {}^2\lambda_{j_2}, {}^2\mu_{i_2} \end{array}\right\} \xrightarrow{\mathbf{H}({}^1s,{}^2s)} \left\{ \mathbf{w}_{j_1,j_2}, \mathbf{v}_{i_1,i_2} \right.$$

${}^1s$ \ ${}^2s$	${}^2\lambda_{1,\cdots,k_2}$	${}^2\mu_{1,\cdots,q_2}$
${}^1\lambda_{1,\cdots,k_1}$	$\mathbf{W}_{k_1,k_2}$	ϕ_{cr}
${}^1\mu_{1,\cdots,q_1}$	ϕ_{rc}	$\mathbf{V}_{q_1,q_2}$

Tensors (2-D) tab$_2$



Forewords

Data (and tensors)

Column / Row data

$$\left.\begin{array}{l} {}^1\lambda_{j_1}, {}^1\mu_{i_1} \\ {}^2\lambda_{j_2}, {}^2\mu_{i_2} \\ {}^3\lambda_{j_3}, {}^3\mu_{i_3} \end{array}\right\} \xrightarrow{\mathbf{H}({}^1s, {}^2s, {}^2s)} \left\{ \mathbf{w}_{j_1,j_2,j_3}, \mathbf{v}_{i_1,i_2,i_3} \right.$$

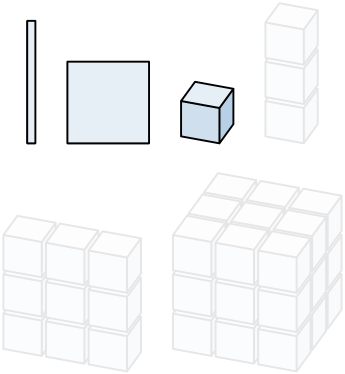
${}^3s = {}^3\lambda_{1,\cdots,k_3}$

${}^1s \backslash {}^2s$	${}^2\lambda_{1,\cdots,k_2}$	${}^2\mu_{1,\cdots,q_2}$
${}^1\lambda_{1,\cdots,k_1}$	$\mathbf{W}_{k_1,k_2,k_3}$	ϕ_{crc}
${}^1\mu_{1,\cdots,q_1}$	ϕ_{rcc}	ϕ_{rrc}

${}^3s = {}^3\mu_{1,\cdots,q_3}$

${}^1s \backslash {}^2s$	${}^2\lambda_{1,\cdots,k_2}$	${}^2\mu_{1,\cdots,q_2}$
${}^1\lambda_{1,\cdots,k_1}$	ϕ_{crr}	ϕ_{crr}
${}^1\mu_{1,\cdots,q_1}$	ϕ_{rcr}	$\mathbf{V}_{q_1,q_2,q_3}$

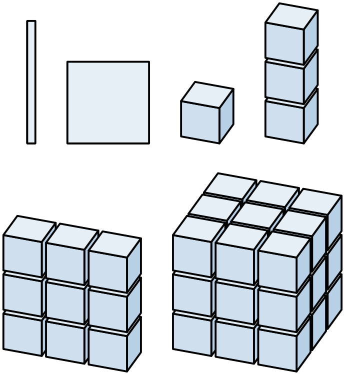
Tensors (3-D) tab$_3$



Column / Row data

$$\left.\begin{array}{c} {}^1\lambda_{j_1}, {}^1\mu_{i_1} \\ {}^2\lambda_{j_2}, {}^2\mu_{i_2} \\ {}^3\lambda_{j_3}, {}^3\mu_{i_3} \\ \vdots \\ {}^n\lambda_{j_n}, {}^n\mu_{i_n} \end{array}\right\} \xrightarrow{\mathbf{H}({}^1s,\cdots,{}^ns)} \left\{ \mathbf{w}_{j_1,\cdots,j_n}, \mathbf{v}_{i_1,\cdots,i_n} \right.$$

Tensors (n-D) tab$_n$



Forewords

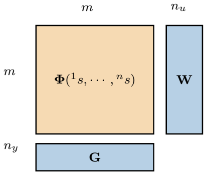
Problem description

Data-driven model approximation

Being given a n-dimensional tensor (data), we seek a multi-variate **rational function** $\hat{\mathbf{H}}$ and **realization** $(\mathbf{G}, \boldsymbol{\Phi}, \mathbf{W})$

$$\hat{\mathbf{H}}({}^1s, {}^2s, \dots, {}^ns) = \mathbf{G}\boldsymbol{\Phi}({}^1s, {}^2s, \dots, {}^ns)^{-1}\mathbf{W} \in \mathbb{C}$$

that interpolates the data.



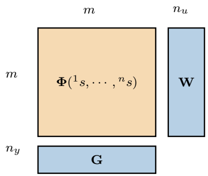
Connection to standard dynamical system realization

A linear-in-state dynamical system parameterized in terms of parameters included in $\mathcal{S} = [{}^2s, \dots, {}^ns]^\top \subset \mathbb{C}^{n-1}$

$$\begin{array}{rcl} \mathbf{E}(\mathcal{S})\dot{\mathbf{x}}(t;\mathcal{S}) & = & \mathbf{A}(\mathcal{S})\mathbf{x}(t;\mathcal{S}) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t;\mathcal{S}) & = & \mathbf{C}\mathbf{x}(t;\mathcal{S}) \end{array}$$

equivalently

$$\hat{\mathbf{H}}({}^1s, {}^2s, \dots, {}^ns) = \mathbf{C}(\mathcal{S})\left[{}^1s\mathbf{E}(\mathcal{S}) - \mathbf{A}(\mathcal{S})\right]^{-1}\mathbf{B} \in \mathbb{C}.$$



Data-driven model approximation

Being given a n-dimensional tensor (data), we seek a multi-variate **rational function** $\hat{\mathbf{H}}$ and **realization** $(\mathbf{G}, \mathbf{\Phi}, \mathbf{W})$

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equivalently

$$\hat{\mathbf{H}}({}^1s, {}^2s, \ldots, {}^ns) = \mathbf{C}(\mathcal{S}) \left[{}^1s\mathbf{E}(\mathcal{S}) - \mathbf{A}(\mathcal{S})\right]^{-1} \mathbf{B} \in \mathbb{C}.$$

Forewords

Where we stand (some references)

1-D Two-sided Loewner
$\Rightarrow$ (interpolation) barycentric form
$\Rightarrow$ realization minimality
$\Rightarrow$ direct algorithm

1-D One-sided Loewner
$\Rightarrow$ (interpolation) barycentric form
$\Rightarrow$ direct algorithm

1-D AAA (Adaptive Anderson Antoulas - one-sided)
$\Rightarrow$ (mixed interpolation LS) barycentric form
$\Rightarrow$ iterative algorithm

2-D Parametric one-sided Loewner
$\Rightarrow$ (interpolation) barycentric form
$\Rightarrow$ realization (non-minimal)
$\Rightarrow$ direct algorithm

3-D Parametric AAA
$\Rightarrow$ (mixed interpolation LS) barycentric form
$\Rightarrow$ iterative algorithm

$>$3-D **few results**

J-P. Berrut and N. Trefethen, *"Barycentric Lagrange Interpolation"*, SIAM Review, 46(3), 2004.

A.J. Mayo and A.C. Antoulas, *"A framework for the solution of the generalized realisation problem"*, LAA, 425(2-3), 2007.

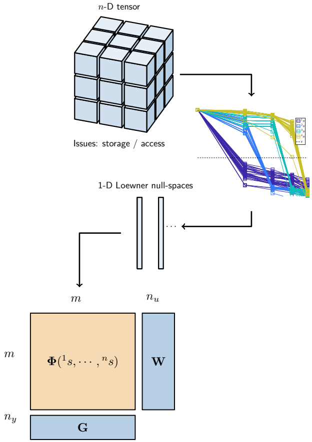
A.C. Ionita and A.C. Antoulas, *"Data-Driven Parametrized Model Reduction in the Loewner Framework"*, SIAM Journal on Scientific Computing, 36(3), 2014.

T. Vojkovic, D. Quero, C. P-V and P. Vuillemin, *"Low-Order Parametric State-Space Modeling of MIMO Systems in the Loewner Framework"*, SIAM Journal on Applied Dynamical Systems, 22(4), 2023.

A.C. Rodriguez, L. Balicki and S. Gugercin, *"The p-AAA algorithm for data driven modeling of parametric dynamical systems"*, SIAM Journal on Scientific Computing, 45(3), 2023.

List of contributions

- n-D tensor data to n-D Loewner matrix $\mathbb{L}_n$
- n-D recursive Sylvester equations
- n-variable transfer functions
- n-variable generalized realization
- Taming the curse of dimensionality
  - in computation effort (`flop`)
  - in storage needs (`Bytes`)
  - in accuracy
- n-variable **decoupling**
  - **KST** formulation for rational functions
  - connection with **KAN**



Content

Multi-variate data, function & Loewner matrix

1-D case

$$\left\{\begin{array}{lll} P_c^{(1)} & := & \left\{\left({}^1\lambda_{j_1}; \mathbf{w}_{j_1}\right),\ j_1 = 1, \dots, k_1\right\} \\ P_r^{(1)} & := & \left\{\left({}^1\mu_{i_1}; \mathbf{v}_{i_1}\right),\ i_1 = 1, \dots, q_1\right\} \end{array}\right.$$

Loewner matrix

$$\mathbb{L}_1 \in \mathbb{C}^{q_1 \times k_1}$$

$$(\mathbb{L}_1)_{i_1,j_1} = \frac{\mathbf{v}_{i_1} - \mathbf{w}_{j_1}}{{}^1\mu_{i_1} - {}^1\lambda_{j_1}}$$

$$\mathbf{M}_1 \mathbb{L}_1 - \mathbb{L}_1 \mathbf{\Lambda}_1 = \mathbb{V}_1 \mathbf{R}_1 - \mathbf{L}_1 \mathbb{W}_1$$

Lagrangian form

$$\mathbf{g}({}^1 s) = \frac{\sum_{j_1=1}^{k_1} \frac{c_{j_1} \mathbf{w}_{j_1}}{{}^1 s - {}^1\lambda_{j_1}}}{\sum_{j_1=1}^{k_1} \frac{c_{j_1}}{{}^1 s - {}^1\lambda_{j_1}}}$$

Null space

$$\textbf{span}\ (\mathbf{c}_1) = \mathcal{N}(\mathbb{L}_1)$$

$$\mathbf{c}_1 = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{k_1} \end{bmatrix} \in \mathbb{C}^{k_1}$$

Multi-variate data, function & Loewner matrix

1-D case (example)

Data generated from $\mathbf{H}({}^1s) = \mathbf{H}(s) = (s^2+4)/(s+1)$ of complexity (2)

$$\left.\begin{array}{rcl} {}^1\lambda_{j_1} & = & [1,3,5] \\ {}^1\mu_{i_1} & = & [2,4,6,8] \end{array}\right\} \xrightarrow{\mathbf{H}} \left\{\begin{array}{rcl} \mathbf{w}_{j_1} & = & [5/2, 13/4, 29/6] \\ \mathbf{v}_{i_1} & = & [8/3, 4, 40/7, 68/9] \end{array}\right.$$

Loewner matrix

$$\mathbb{L}_1 = \begin{bmatrix} \frac{1}{6} & \frac{7}{12} & \frac{13}{18} \\ \frac{1}{2} & \frac{3}{4} & \frac{5}{6} \\ \frac{9}{14} & \frac{23}{28} & \frac{37}{42} \\ \frac{13}{18} & \frac{31}{36} & \frac{49}{54} \end{bmatrix}$$

Null space

$$\mathbf{c}_1 = \begin{bmatrix} \frac{1}{3} \\ -\frac{4}{3} \\ 1 \end{bmatrix}$$

Lagrangian form

$$\mathbf{g}(s) = \frac{\frac{5}{6\,(s-1)} - \frac{13}{3\,(s-3)} + \frac{29}{6\,(s-5)}}{\frac{1}{3\,(s-1)} - \frac{4}{3\,(s-3)} + \frac{1}{s-5}} = \mathbf{H}(s)$$

Multi-variate data, function & Loewner matrix

2-D case

$$\begin{cases} P_c^{(2)} & := \left\{ ({}^1\lambda_{j_1}, {}^2\lambda_{j_2}; \mathbf{w}_{j_1,j_2}),\ j_1 = 1,\dots,k_1 \quad j_2 = 1,\dots,k_2 \right\} \\ P_r^{(2)} & := \left\{ ({}^1\mu_{i_1}, {}^2\mu_{i_2}; \mathbf{v}_{i_1,i_2}),\ i_1 = 1,\dots,q_1 \quad i_2 = 1,\dots,q_2 \right\} \end{cases}$$

Loewner matrix

$$\mathbb{L}_2 \in \mathbb{C}^{q_1 q_2 \times k_1 k_2}$$

$$\ell_{j_1,j_2}^{i_1,i_2} = \frac{\mathbf{v}_{i_1,i_2} - \mathbf{w}_{j_1,j_2}}{({}^1\mu_{i_1} - {}^1\lambda_{j_1})({}^2\mu_{i_2} - {}^2\lambda_{j_2})}$$

$$\begin{cases} \mathbf{M}_2\mathbb{X} - \mathbb{X}\mathbf{\Lambda}_2 & = & \mathbb{V}_2\mathbf{R}_2 - \mathbf{L}_2\mathbb{W}_2 \\ \mathbf{M}_1\mathbb{L}_2 - \mathbb{L}_2\mathbf{\Lambda}_1 & = & \mathbb{X} \end{cases}$$

Lagrangian form

$$\mathbf{g}({}^1s, {}^2s) = \frac{\sum_{j_1=1}^{k_1} \sum_{j_2=1}^{k_2} \frac{c_{j_1,j_2}\mathbf{w}_{j_1,j_2}}{({}^1s - {}^1\lambda_{j_1})({}^2s - {}^2\lambda_{j_2})}}{\sum_{j_1=1}^{k_1} \sum_{j_2=1}^{k_2} \frac{c_{j_1,j_2}}{({}^1s - {}^1\lambda_{j_1})({}^2s - {}^2\lambda_{j_2})}}$$

Null space

$$\textbf{span}\ (\mathbf{c}_2) = \mathcal{N}(\mathbb{L}_2)$$

$$\mathbf{c}_2 = \left[\begin{array}{c} c_{1,1} \\ \vdots \\ c_{1,k_2} \\ \hline \vdots \\ \hline c_{k_1,1} \\ \vdots \\ c_{k_1,k_2} \end{array}\right] \in \mathbb{C}^{k_1 k_2}$$

Multi-variate data, function & Loewner matrix

2-D case (example)

Data generated from $\mathbf{H}({}^1s, {}^2s) = \mathbf{H}(s,t) = (s^2t)/(s-t+1)$ of complexity $(2,1)$

$$\left.\begin{array}{rcl} {}^1\lambda_{j_1} &=& [1,3,5] \\ {}^1\mu_{i_1} &=& [0,2,4] \\ {}^2\lambda_{j_2} &=& [-1,-3] \\ {}^2\mu_{i_2} &=& [-2,-4] \end{array}\right\} \xrightarrow{\mathbf{H}} \left[\begin{array}{cc|cc} -\frac{1}{3} & -\frac{3}{5} & -\frac{1}{2} & -\frac{2}{3} \\ -\frac{9}{5} & -\frac{27}{7} & -3 & -\frac{9}{2} \\ -\frac{25}{7} & -\frac{25}{3} & -\frac{25}{4} & -10 \\ \hline 0 & 0 & 0 & 0 \\ -1 & -2 & -\frac{8}{5} & -\frac{16}{7} \\ -\frac{8}{3} & -6 & -\frac{32}{7} & -\frac{64}{9} \end{array}\right]$$

Loewner matrix

$$\mathbb{L}_2 = \left[\begin{array}{cc|cc|cc} \frac{1}{3} & -\frac{3}{5} & \frac{3}{5} & -\frac{9}{7} & \frac{5}{7} & -\frac{5}{3} \\ \frac{1}{9} & \frac{3}{5} & \frac{1}{5} & \frac{9}{7} & \frac{5}{21} & \frac{5}{3} \\ \frac{19}{15} & -1 & \frac{1}{5} & -\frac{79}{35} & \frac{23}{35} & -\frac{101}{45} \\ \hline \frac{41}{63} & \frac{59}{35} & -\frac{17}{105} & \frac{11}{7} & \frac{1}{7} & \frac{127}{63} \\ \frac{89}{63} & -\frac{139}{105} & \frac{97}{35} & -\frac{5}{7} & -1 & -\frac{79}{21} \\ \frac{61}{81} & \frac{293}{135} & \frac{239}{135} & \frac{205}{63} & -\frac{223}{189} & \frac{11}{9} \end{array}\right]$$

Null space

$$\mathbf{c}_2 = \left[\begin{array}{c} -\frac{1}{3} \\ \frac{5}{9} \\ \hline \frac{10}{9} \\ -\frac{14}{9} \\ \hline -\frac{7}{9} \\ 1 \end{array}\right]$$

Multi-variate data, function & Loewner matrix

2-D case (example)

Data generated from $\mathbf{H}({}^1s, {}^2s) = \mathbf{H}(s,t) = (s^2t)/(s-t+1)$ of complexity $(2,1)$

$$\left.\begin{array}{rcl} {}^1\lambda_{j_1} & = & [1,3,5] \\ {}^1\mu_{i_1} & = & [0,2,4] \\ {}^2\lambda_{j_2} & = & [-1,-3] \\ {}^2\mu_{i_2} & = & [-2,-4] \end{array}\right\} \xrightarrow{\mathbf{H}} \left[\begin{array}{cc|cc} -\frac{1}{3} & -\frac{3}{5} & -\frac{1}{2} & -\frac{2}{3} \\ -\frac{9}{5} & -\frac{27}{7} & -3 & -\frac{9}{2} \\ -\frac{25}{7} & -\frac{25}{3} & -\frac{25}{4} & -10 \\ \hline 0 & 0 & 0 & 0 \\ -1 & -2 & -\frac{8}{5} & -\frac{16}{7} \\ -\frac{8}{3} & -6 & -\frac{32}{7} & -\frac{64}{9} \end{array}\right]$$

Lagrangian form

$$\mathbf{g}(s,t) = -\frac{\frac{1}{9(s-1)(t+1)} - \frac{1}{3(s-1)(t+3)} - \frac{2}{(s-3)(t+1)} + \frac{6}{(s-3)(t+3)} + \frac{25}{9(s-5)(t+1)} - \frac{25}{3(s-5)(t+3)}}{\frac{1}{3(s-1)(t+1)} - \frac{5}{9(s-1)(t+3)} - \frac{10}{9(s-3)(t+1)} + \frac{14}{9(s-3)(t+3)} + \frac{7}{9(s-5)(t+1)} - \frac{1}{(s-5)(t+3)}} = \mathbf{H}(s,t)$$

Multi-variate data, function & Loewner matrix

n-D case

$$\begin{cases} P_c^{(n)} := \left\{ ({}^1\lambda_{j_1}, {}^2\lambda_{j_2}, \cdots, {}^n\lambda_{j_n}; \mathbf{w}_{j_1,j_2,\cdots,j_n}),\ j_l = 1,\dots,k_l, \quad l = 1,\cdots,n \right\} \\ P_r^{(n)} := \left\{ ({}^1\mu_{i_1}, {}^2\mu_{i_2}, \cdots, {}^n\mu_{i_n}; \mathbf{v}_{i_1,i_2,\cdots,i_n}),\ i_l = 1,\dots,q_l, \quad l = 1,\cdots,n \right\} \end{cases}$$

Loewner matrix

$$\mathbb{L}_n \in \mathbb{C}^{q_1 q_2 \cdots q_n \times k_1 k_2 \cdots k_n}$$

$$\ell_{j_1,j_2,\cdots,j_n}^{i_1,i_2,\cdots,i_n} = \frac{\mathbf{v}_{i_1,i_2,\cdots,i_n} - \mathbf{w}_{j_1,j_2,\cdots,j_n}}{({}^1\mu_{i_1} - {}^1\lambda_{j_1}) \cdots ({}^n\mu_{i_n} - {}^n\lambda_{j_n})}$$

$$\begin{cases} \mathbf{M}_n \mathbb{X}_1 - \mathbb{X}_1 \mathbf{\Lambda}_n & = & \mathbb{V}_n \mathbf{R}_n - \mathbf{L}_n \mathbb{W}_n, \\ \mathbf{M}_{n-1} \mathbb{X}_2 - \mathbb{X}_2 \mathbf{\Lambda}_{n-1} & = & \mathbb{X}_1, \\ & \cdots & \\ \mathbf{M}_1 \mathbb{L}_n - \mathbb{L}_n \mathbf{\Lambda}_1 & = & \mathbb{X}_{n-1}. \end{cases}$$

Lagrangian form

$$\mathbf{g}({}^1s, \cdots, {}^ns) = \frac{\sum_{j_1=1}^{k_1} \cdots \sum_{j_n=1}^{k_n} \frac{c_{j_1,\cdots,j_n} \mathbf{w}_{j_1,\cdots,j_n}}{({}^1s - {}^1\lambda_{j_1}) \cdots ({}^ns - {}^n\lambda_{j_n})}}{\sum_{j_1=1}^{k_1} \cdots \sum_{j_n=1}^{k_n} \frac{c_{j_1,\cdots,j_n}}{({}^1s - {}^1\lambda_{j_1}) \cdots ({}^ns - {}^n\lambda_{j_n})}}$$

Null space

$$\textbf{span}\ (\mathbf{c}_n) = \mathcal{N}(\mathbb{L}_n)$$

$$\mathbf{c}_n = \left[\begin{array}{c} c_{1,\cdots,1} \\ \vdots \\ c_{1,\cdots,k_n} \\ \hline \vdots \\ \hline c_{k_1,\cdots,1} \\ \vdots \\ c_{k_1,\cdots,k_n} \end{array} \right] \in \mathbb{C}^{k_1 \cdots k_n}$$

Content

Multi-variate realization

1-D case (example cont'd)

Data generated from $\mathbf{H}({}^1s) = \mathbf{H}(s) = (s^2+4)/(s+1)$ of complexity (2)

$$\mathbf{g}(s) = \frac{\frac{5}{6\,(s-1)} - \frac{13}{3\,(s-3)} + \frac{29}{6\,(s-5)}}{\frac{1}{3\,(s-1)} - \frac{4}{3\,(s-3)} + \frac{1}{s-5}}$$

Lagrangian realization $\hat{\mathbf{H}}(s) = \mathbf{W}\boldsymbol{\Phi}(s)^{-1}\mathbf{G}$

$$\boldsymbol{\Phi}(s) = \left[\begin{array}{ccc} s-1 & 3-s & 0 \\ s-1 & 0 & 5-s \\ \hdashline -\frac{1}{3} & \frac{4}{3} & -1 \end{array}\right]$$

$$\mathbf{W} = \left[\begin{array}{ccc} \frac{5}{6} & -\frac{13}{3} & \frac{29}{6} \end{array}\right]$$

$$\mathbf{G}^\top = \left[\begin{array}{cc:c} 0 & 0 & -1 \end{array}\right]$$

Multi-variate realization

2-D case (example cont'd)

Data generated from $\mathbf{H}({}^1s, {}^2s) = \mathbf{H}(s,t) = (s^2t)/(s-t+1)$ of complexity $(2,1)$

$$\mathbf{g}(s,t) = -\frac{\frac{1}{9(s-1)(t+1)} - \frac{1}{3(s-1)(t+3)} - \frac{2}{(s-3)(t+1)} + \frac{6}{(s-3)(t+3)} + \frac{25}{9(s-5)(t+1)} - \frac{25}{3(s-5)(t+3)}}{\frac{1}{3(s-1)(t+1)} - \frac{5}{9(s-1)(t+3)} - \frac{10}{9(s-3)(t+1)} + \frac{14}{9(s-3)(t+3)} + \frac{7}{9(s-5)(t+1)} - \frac{1}{(s-5)(t+3)}}$$

Lagrangian realization $\hat{\mathbf{H}}(s,t) = \mathbf{W}\boldsymbol{\Phi}(s,t)^{-1}\mathbf{G}$

$$\boldsymbol{\Phi}(s,t) = \left[\begin{array}{ccc|c|cc} s-1 & 3-s & 0 & 0 & 0 & 0 \\ s-1 & 0 & 5-s & 0 & 0 & 0 \\ \hline -\frac{1}{3} & \frac{10}{9} & -\frac{7}{9} & t+1 & 0 & 0 \\ \frac{5}{9} & -\frac{14}{9} & 1 & -t-3 & 0 & 0 \\ \hline \frac{1}{9} & -2 & \frac{25}{9} & 0 & t+1 & \frac{1}{2} \\ -\frac{1}{3} & 6 & -\frac{25}{3} & 0 & -t-3 & -\frac{1}{2} \end{array}\right]$$

$$\mathbf{W} = \left[\begin{array}{ccccc|c} 0 & 0 & 0 & 0 & 0 & -1 \end{array}\right]$$

$$\mathbf{G}^\top = \left[\begin{array}{cc|cc|cc} 0 & 0 & 1/2 & -1/2 & 0 & 0 \end{array}\right]$$

$\rightarrow$ (3,3) block is unimodular !

Multi-variate realization

2-D case (example cont'd)

Data generated from $\mathbf{H}({}^1s,{}^2s)=\mathbf{H}(s,t)=(s^2t)/(s-t+1)$ of complexity $(2,1)$

$$\mathbf{g}(s,t)=-\frac{\frac{1}{9(s-1)(t+1)}-\frac{1}{3(s-1)(t+3)}-\frac{2}{(s-3)(t+1)}+\frac{6}{(s-3)(t+3)}+\frac{25}{9(s-5)(t+1)}-\frac{25}{3(s-5)(t+3)}}{\frac{1}{3(s-1)(t+1)}-\frac{5}{9(s-1)(t+3)}-\frac{10}{9(s-3)(t+1)}+\frac{14}{9(s-3)(t+3)}+\frac{7}{9(s-5)(t+1)}-\frac{1}{(s-5)(t+3)}}$$

Lagrangian realization $\hat{\mathbf{H}}_{\mathsf{c}}(s,t)=\mathbf{W}_{\mathsf{c}}(t)\mathbf{\Phi}_{\mathsf{c}}(s,t)^{-1}\mathbf{G}_{\mathsf{c}}$

$$\mathbf{\Phi}_{\mathsf{c}}(s,t)=\left[\begin{array}{ccc|c} s-1 & 3-s & 0 & 0\\ s-1 & 0 & 5-s & 0\\ \hline -\frac{1}{3} & \frac{10}{9} & -\frac{7}{9} & t+1\\ \frac{5}{9} & -\frac{14}{9} & 1 & -t-3 \end{array}\right]$$

$$\mathbf{W}_{\mathsf{c}}(t)=\left[\begin{array}{ccc|c} -\frac{2\,t}{9} & 4\,t & -\frac{50\,t}{9} & 0\end{array}\right]$$

$$\mathbf{G}_{\mathsf{c}}^{\top}=\left[\begin{array}{cc|cc} 0 & 0 & \frac{1}{2} & -\frac{1}{2}\end{array}\right]$$

Multi-variate realization

Generalized n-D Lagrangian realization

$$\mathbf{g}({}^1s, {}^2s, \cdots, {}^ns) = \frac{\sum_{j_1=1}^{k_1} \sum_{j_2=1}^{k_2} \cdots \sum_{j_n=1}^{k_n} \frac{c_{j_1,j_2,\cdots,j_n} \mathbf{w}_{j_1,j_2,\cdots,j_n}}{({}^1s - {}^1\lambda_{j_1})({}^2s - {}^2\lambda_{j_2})\cdots({}^ns - {}^n\lambda_{j_n})}}{\sum_{j_1=1}^{k_1} \sum_{j_2=1}^{k_2} \cdots \sum_{j_n=1}^{k_n} \frac{c_{j_1,j_2,\cdots,j_n}}{({}^1s - {}^1\lambda_{j_1})({}^2s - {}^2\lambda_{j_2})\cdots({}^ns - {}^n\lambda_{j_n})}},$$

Theorem: n-D Lagrangian realization

A $2\ell + \kappa - 1 = m$-th order realization $(\mathbf{G}, \mathbf{\Phi}, \mathbf{W})$ of the multi-variate function $\hat{\mathbf{H}}$ in barycentric form, satisfying $\hat{\mathbf{H}}({}^1s, \cdots, {}^ns) = \mathbf{W}\mathbf{\Phi}({}^1s, {}^2s, \cdots, {}^ns)^{-1}\mathbf{G}$, is given by,

$$\begin{aligned}
\mathbf{\Phi}({}^1s, \cdots, {}^ns) &= \left[\begin{array}{c|c|c} \mathbf{\Gamma}(1:\kappa-1,:) & \mathbf{0}_{\kappa-1,\ell-1} & \mathbf{0}_{\kappa-1,\ell} \\ \hline \mathbb{A}^{\text{Lag}} & \mathbf{\Delta}(1:\ell-1,:)^\top & \mathbf{0}_{\ell,\ell} \\ \hline \mathbb{B}^{\text{Lag}} & \mathbf{0}_{\ell,\ell-1} & \mathbf{\Delta}^\top \end{array}\right] \in \mathbb{C}^{m\times m} \\
\mathbf{G} &= \left[\begin{array}{c} \mathbf{0}_{\kappa-1,1} \\ \hline \mathbf{\Delta}(\ell,:)^\top \\ \hline \mathbf{0}_{\ell,1} \end{array}\right] \in \mathbb{C}^{m\times 1} \\
\mathbf{W} &= \left[\begin{array}{c|c|c} \mathbf{0}_{1,\kappa} & \mathbf{0}_{1,\ell-1} & -\mathbf{e}_\ell^\top \end{array}\right] \in \mathbb{C}^{1\times m}
\end{aligned}$$

where $\mathbb{A}^{\text{Lag}}, \mathbb{B}^{\text{Lag}} \in \mathbb{C}^{\ell\times\kappa}$ are appropriately chosen, according to the chosen pseudo-companion basis.

Multi-variate realization

Generalized n-D Lagrangian realization (focus on left / right variable sets)

$$\mathbf{\Phi}({}^1s,\cdots,{}^ns) = \left[\begin{array}{c:c:c} \mathbf{\Gamma}(1:\kappa-1,:) & \mathbf{0}_{\kappa-1,\ell-1} & \mathbf{0}_{\kappa-1,\ell} \\ \hdashline \mathbb{A}^{\mathrm{Lag}} & \mathbf{\Delta}(1:\ell-1,:)^\top & \mathbf{0}_{\ell,\ell} \\ \hdashline \mathbb{B}^{\mathrm{Lag}} & \mathbf{0}_{\ell,\ell-1} & \mathbf{\Delta}^\top \end{array}\right] \in \mathbb{C}^{m\times m}$$

$$\begin{aligned}
\mathbf{\Gamma} &= {}^1\mathbb{X}^{\mathrm{Lag}} \otimes {}^2\mathbb{X}^{\mathrm{Lag}} \otimes \cdots \otimes {}^k\mathbb{X}^{\mathrm{Lag}} \in \mathbb{C}^{\kappa\times\kappa}[{}^1s,\cdots,{}^ks] \\
\mathbf{\Delta} &= {}^{k+1}\mathbb{X}^{\mathrm{Lag}} \otimes {}^{k+2}\mathbb{X}^{\mathrm{Lag}} \otimes \cdots \otimes {}^n\mathbb{X}^{\mathrm{Lag}} \in \mathbb{C}^{\ell\times\ell}[{}^{k+1}s,\cdots,{}^ns] \\
{}^j\mathbb{X}^{\mathrm{Lag}} &= \left[\begin{array}{ccccc} {}^j\mathbf{x}_1 & -{}^j\mathbf{x}_2 & 0 & \cdots & 0 \\ {}^j\mathbf{x}_1 & 0 & -{}^j\mathbf{x}_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ {}^j\mathbf{x}_1 & 0 & \cdots & 0 & -{}^j\mathbf{x}_{n_j} \\ \hline {}^jq_1 & {}^jq_2 & \cdots & {}^jq_{n_j-1} & {}^jq_{n_j} \end{array}\right] \in \mathbb{C}^{n_j\times n_j} \\
{}^j\mathbf{x}_i &= {}^js - {}^j\lambda_i
\end{aligned}$$

Facts

- Left / right variables splitting

$$\Gamma \text{ and } \Delta$$

- ${}^j\mathbb{X}^{\mathrm{Lag}}$ is unimodular, i.e.

$$\det({}^j\mathbb{X}^{\mathrm{Lag}}) = 1$$

- ...so are Γ and Δ

Multi-variate realization

Generalized n-D Lagrangian realization (focus on barycentric weights $\mathbb{A}^{\mathrm{Lag}}$ and $\mathbb{B}^{\mathrm{Lag}}$)

$$\mathbf{\Phi}({}^1s,\cdots,{}^ns) = \left[\begin{array}{c:c:c} \mathbf{\Gamma}(1:\kappa-1,:) & \mathbf{0}_{\kappa-1,\ell-1} & \mathbf{0}_{\kappa-1,\ell} \\ \hdashline \mathbb{A}^{\mathrm{Lag}} & \mathbf{\Delta}(1:\ell-1,:)^\top & \mathbf{0}_{\ell,\ell} \\ \hdashline \mathbb{B}^{\mathrm{Lag}} & \mathbf{0}_{\ell,\ell-1} & \mathbf{\Delta}^\top \end{array}\right] \in \mathbb{C}^{m\times m}$$

$$\begin{array}{rcl} \mathbb{A}^{\mathrm{Lag}} & = & \begin{bmatrix} \alpha_{1,1} & \alpha_{1,2} & \cdots & \alpha_{1,m+1} \\ \alpha_{2,1} & \alpha_{2,2} & \cdots & \alpha_{2,m+1} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n+1,1} & \alpha_{n+1,2} & \cdots & \alpha_{n+1,m+1} \end{bmatrix} \\ \mathbf{c}_n & = & \mathrm{vec}\left(\mathbb{A}^{\mathrm{Lag}}\right) \end{array}$$

Facts

- $\mathbb{A}^{\mathrm{Lag}}$ is simply some rearrangement of $\mathcal{N}(\mathbb{L}_n) = \mathbf{c}_n$
- $\mathbb{B}^{\mathrm{Lag}}$ follows

Multi-variate realization

Generalized n-D Lagrangian realization (control the complexity)

$$\mathbf{\Phi}({}^1s,\cdots,{}^ns) = \left[\begin{array}{c|c|c} \mathbf{\Gamma}(1:\kappa-1,:) & \mathbf{0}_{\kappa-1,\ell-1} & \mathbf{0}_{\kappa-1,\ell} \\ \hdashline \mathbb{A}^{\text{Lag}} & \mathbf{\Delta}(1:\ell-1,:)^\top & \mathbf{0}_{\ell,\ell} \\ \hdashline \mathbb{B}^{\text{Lag}} & \mathbf{0}_{\ell,\ell-1} & \mathbf{\Delta}^\top \end{array}\right] \in \mathbb{C}^{m\times m}$$

$$\begin{aligned} m &= 2\ell+\kappa-1 \\ \kappa &= \textstyle\prod_{j=1}^{k} n_j \\ \ell &= \textstyle\prod_{j=k+1}^{n} n_j \end{aligned}$$

Facts

- Γ gathers the first group of parameters ${}^1s,\cdots,{}^ks$
- $\triangle$ gathers the second group of parameters ${}^{k+1}s,\cdots,{}^ns$

Complexity

Re-ordering allows complexity control, e.g. according to the order of each variable ${}^js$

Multi-variate realization

3-D case (example)

Data generated from $\mathbf{H}({}^1s, {}^2s, {}^3s) = \mathbf{H}(s,t,p) = (s+pt)/(p^2+s+t)$ of complexity $(1,1,2)$

$$
\begin{array}{rcl}
\mathbf{c}_3^\top & = & \left[\begin{array}{ccc|ccc|ccc|ccc} \frac{1}{2} & -\frac{39}{28} & \frac{13}{14} & -\frac{15}{28} & \frac{41}{28} & -\frac{27}{28} & -\frac{15}{28} & \frac{41}{28} & -\frac{27}{28} & \frac{4}{7} & -\frac{43}{28} & 1 \end{array}\right] \\
\mathbb{W}_3 & = & \left[\begin{array}{ccc|ccc|ccc|ccc} \frac{1}{4} & \frac{8}{39} & \frac{9}{52} & \frac{17}{30} & \frac{20}{41} & \frac{23}{54} & \frac{3}{10} & \frac{10}{41} & \frac{11}{54} & \frac{19}{32} & \frac{22}{43} & \frac{25}{56} \end{array}\right]
\end{array}
$$

Arrangement #1

$(s)-(t,p)$, one obtains a realization of dimension $m=13$:

$$\kappa = 2 \text{ and } \ell = 2\times 3$$

$$\mathbf{\Delta}(s) = {}^1\mathbb{X}^{\mathrm{Lag}}(s)$$

$$\mathbf{\Gamma}(t,p) = {}^2\mathbb{X}^{\mathrm{Lag}}(t) \otimes {}^3\mathbb{X}^{\mathrm{Lag}}(p)$$

Arrangement #2

$(s,t)-(p)$, one obtains a realization of dimension $m=9$:

$$\kappa = 2\times 2 \text{ and } \ell = 3$$

$$\mathbf{\Delta}(s,t) = {}^1\mathbb{X}^{\mathrm{Lag}}(s) \otimes {}^2\mathbb{X}^{\mathrm{Lag}}(t)$$

$$\mathbf{\Gamma}(p) = {}^3\mathbb{X}^{\mathrm{Lag}}(p)$$

Multi-variate realization

3-D case (example)

Data generated from $\mathbf{H}({}^1s, {}^2s, {}^3s) = \mathbf{H}(s,t,p) = (s+pt)/(p^2+s+t)$ of complexity $(1,1,2)$

$$
\begin{array}{rcl}
\mathbf{c}_3^\top & = & \left[\begin{array}{ccc|ccc|ccc|ccc} \frac{1}{2} & -\frac{39}{28} & \frac{13}{14} & -\frac{15}{28} & \frac{41}{28} & -\frac{27}{28} & -\frac{15}{28} & \frac{41}{28} & -\frac{27}{28} & \frac{4}{7} & -\frac{43}{28} & 1 \end{array}\right] \\
\mathbb{W}_3 & = & \left[\begin{array}{ccc|ccc|ccc|ccc} \frac{1}{4} & \frac{8}{39} & \frac{9}{52} & \frac{17}{30} & \frac{20}{41} & \frac{23}{54} & \frac{3}{10} & \frac{10}{41} & \frac{11}{54} & \frac{19}{32} & \frac{22}{43} & \frac{25}{56} \end{array}\right]
\end{array}
$$

$$
\mathbf{\Phi} = \left[\begin{array}{cccc:cc:ccc}
(s-2)\ (t-1) & -(s-2)\ (t-3) & -(t-1)\ (s-4) & (s-4)\ (t-3) & 0 & 0 & 0 & 0 & 0 \\
1-\frac{s}{2} & \frac{s}{2}-1 & \frac{s}{2}-2 & 2-\frac{s}{2} & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2}-\frac{t}{2} & \frac{t}{2}-\frac{3}{2} & \frac{t}{2}-\frac{1}{2} & \frac{3}{2}-\frac{t}{2} & 0 & 0 & 0 & 0 & 0 \\
\hdashline
\frac{1}{2} & -\frac{15}{28} & -\frac{15}{28} & \frac{4}{7} & p-5 & p-5 & 0 & 0 & 0 \\
-\frac{39}{28} & \frac{41}{28} & \frac{41}{28} & -\frac{43}{28} & 6-p & 0 & 0 & 0 & 0 \\
\frac{13}{14} & -\frac{27}{28} & -\frac{27}{28} & 1 & 0 & 7-p & 0 & 0 & 0 \\
\hdashline
\frac{1}{8} & -\frac{17}{56} & -\frac{9}{56} & \frac{19}{56} & 0 & 0 & p-5 & p-5 & \frac{1}{2} \\
-\frac{2}{7} & \frac{5}{7} & \frac{5}{14} & -\frac{11}{14} & 0 & 0 & 6-p & 0 & -1 \\
\frac{9}{56} & -\frac{23}{56} & -\frac{11}{56} & \frac{25}{56} & 0 & 0 & 0 & 7-p & \frac{1}{2}
\end{array}\right]
$$

$\mathbf{W} = -\mathbf{e}_9^\top$ and $\mathbf{G}^\top = \left[\begin{array}{c:ccc:c} \mathbf{0}_{1,3} & 1/2 & -1 & 1/2 & \mathbf{0}_{1,3} \end{array}\right]$

Multi-variate realization

3-D case (example)

Data generated from $\mathbf{H}({}^1s, {}^2s, {}^3s) = \mathbf{H}(s,t,p) = (s+pt)/(p^2+s+t)$ of complexity $(1,1,2)$

$$
\begin{aligned}
\mathbf{c}_3^\top &= \left[\begin{array}{ccc|ccc|ccc|ccc} \frac{1}{2} & -\frac{39}{28} & \frac{13}{14} & -\frac{15}{28} & \frac{41}{28} & -\frac{27}{28} & -\frac{15}{28} & \frac{41}{28} & -\frac{27}{28} & \frac{4}{7} & -\frac{43}{28} & 1 \end{array}\right]\\
\mathbb{W}_3 &= \left[\begin{array}{ccc|ccc|ccc|ccc} \frac{1}{4} & \frac{8}{39} & \frac{9}{52} & \frac{17}{30} & \frac{20}{41} & \frac{23}{54} & \frac{3}{10} & \frac{10}{41} & \frac{11}{54} & \frac{19}{32} & \frac{22}{43} & \frac{25}{56} \end{array}\right]
\end{aligned}
$$

$$
\mathbf{\Phi}_{\mathsf{c}} = \left[\begin{array}{cccc:cc}
(s-2)\ (t-1) & -(s-2)\ (t-3) & -(t-1)\ (s-4) & (s-4)\ (t-3) & 0 & 0\\
1-\frac{s}{2} & \frac{s}{2}-1 & \frac{s}{2}-2 & 2-\frac{s}{2} & 0 & 0\\
\frac{1}{2}-\frac{t}{2} & \frac{t}{2}-\frac{3}{2} & \frac{t}{2}-\frac{1}{2} & \frac{3}{2}-\frac{t}{2} & 0 & 0\\
\hdashline
\frac{1}{2} & -\frac{15}{28} & -\frac{15}{28} & \frac{4}{7} & p-5 & p-5\\
-\frac{39}{28} & \frac{41}{28} & \frac{41}{28} & -\frac{43}{28} & 6-p & 0\\
\frac{13}{14} & -\frac{27}{28} & -\frac{27}{28} & 1 & 0 & 7-p
\end{array}\right]
$$

$$
\mathbf{W}_{\mathsf{c}}(p) = \left[\begin{array}{cccc:cc} \frac{p}{28}+\frac{1}{14} & -\frac{3\,p}{28}-\frac{1}{14} & -\frac{p}{28}-\frac{1}{7} & \frac{3\,p}{28}+\frac{1}{7} & 0 & 0 \end{array}\right] \text{ and } \mathbf{G}_{\mathsf{c}}^\top = \left[\begin{array}{c:ccc} \mathbf{0}_{1,3} & 1/2 & -1 & 1/2 \end{array}\right]
$$

Multi-variate realization

(Compressed) generalized n-D Lagrangian realization

$$\mathbf{g}({}^1s, {}^2s, \cdots, {}^ns) = \frac{\sum_{j_1=1}^{k_1} \sum_{j_2=1}^{k_2} \cdots \sum_{j_n=1}^{k_n} \frac{c_{j_1,j_2,\cdots,j_n} \mathbf{w}_{j_1,j_2,\cdots,j_n}}{({}^1s - {}^1\lambda_{j_1})({}^2s - {}^2\lambda_{j_2})\cdots({}^ns - {}^n\lambda_{j_n})}}{\sum_{j_1=1}^{k_1} \sum_{j_2=1}^{k_2} \cdots \sum_{j_n=1}^{k_n} \frac{c_{j_1,j_2,\cdots,j_n}}{({}^1s - {}^1\lambda_{j_1})({}^2s - {}^2\lambda_{j_2})\cdots({}^ns - {}^n\lambda_{j_n})}},$$

Theorem: n-D Lagrangian compressed realization

A $\ell + \kappa - 1 = m$-th order realization $(\hat{\mathbf{G}}_c, \hat{\mathbf{\Phi}}_c, \hat{\mathbf{W}}_c)$ of the multi-variate function $\hat{\mathbf{H}}$ in barycentric form, satisfying $\mathbf{H}({}^1s, \cdots, {}^ns) = \hat{\mathbf{W}}_c \hat{\mathbf{\Phi}}_c({}^1s, {}^2s, \cdots, {}^ns)^{-1} \hat{\mathbf{G}}_c({}^{k+1}s, \cdots, {}^ns)$, is given by,

$$\begin{aligned}
\hat{\mathbf{\Phi}}_c({}^1s, \cdots, {}^ns) &= \left[\begin{array}{c|c} \mathbf{\Gamma}(1:\kappa-1,:) & \mathbf{0}_{\kappa-1,\ell-1} \\ \hline \mathbb{A}^{\text{Lag}} & \mathbf{\Delta}(1:\ell-1,:)^\top \end{array}\right] \in \mathbb{C}^{m \times m} \\
\hat{\mathbf{G}}_c({}^{k+1}s, \cdots, {}^ns) &= \left[\begin{array}{c} \mathbf{0}_{\kappa-1,1} \\ \hline \mathbf{\Delta}(\ell,:)^\top \end{array}\right] \in \mathbb{C}^{m \times 1} \\
\hat{\mathbf{W}}_c &= \mathbf{e}_\ell^\top \mathbf{\Delta}^{-\top} \left[\begin{array}{c|c} \mathbb{B}^{\text{Lag}} & \mathbf{0}_{\ell,\ell-1} \end{array}\right] \in \mathbb{C}^{1 \times m}
\end{aligned}$$

where $\mathbb{A}^{\text{Lag}}, \mathbb{B}^{\text{Lag}} \in \mathbb{C}^{\ell \times \kappa}$ are appropriately chosen, according to the chosen pseudo-companion basis.

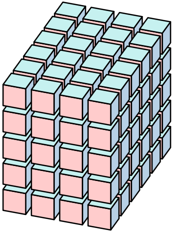
Content

Taming the curse of dimensionality

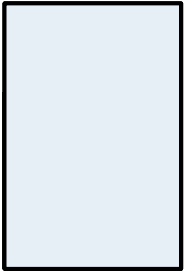
Loewner matrix operator

$$
\begin{array}{ccc}
\mathbb{C}^{k_1} \times \mathbb{C}^{q_1} \times \ldots \times \mathbb{C}^{k_n} \times \mathbb{C}^{q_n} \times \mathbb{C}^{(k_1+q_1)\times\cdots\times(k_n+q_n)} & \longrightarrow & \mathbb{C}^{Q\times K} \\
\left({}^1\lambda_{j_1}, {}^1\mu_{i_1}, \ldots, {}^n\lambda_{j_n}, {}^n\mu_{i_n}, \mathbf{tab}_n\right) & \longmapsto & \mathbb{L}_n
\end{array}
$$

n-D tensor $\mathbf{tab}_n$



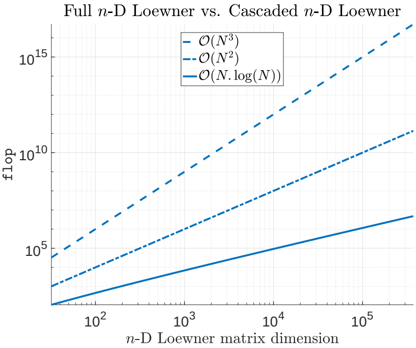
matrix $\mathbb{L}_n$



Taming the curse of dimensionality

Null space `flop` and memory issues

log-log scale



Let (rows) $Q = q_1 q_2 \dots q_n$ and (columns) $K = k_1 k_2 \dots k_n$

$$\mathbb{L}_n \in \mathbb{C}^{Q \times K}$$

Computational issue

Note that $Q \times K$ matrix SVD `flop` estimation is

- QK^2 (if $Q > K$)
- N^3 (if $Q = K = N$)

Storage issue

Note that $Q \times K$ matrix storage estimation is

- in real double
- in complex double

Taming the curse of dimensionality
2-D case (example cont'd, to get the idea)

Data generated from $\mathbf{H}({}^1s, {}^2s) = \mathbf{H}(s,t) = (s^2 t)/(s-t+1)$ of complexity $(2,1)$

${}^1s \backslash {}^2s$	${}^2\lambda_1 = -1$	${}^2\lambda_2 = -3$	${}^2\mu_1 = -2$	${}^2\mu_2 = -4$
${}^1\lambda_1 = 1$	$h_{1,1} = -\frac{1}{3}$	$h_{1,2} = -\frac{3}{5}$	$h_{1,3} = -\frac{1}{2}$	$h_{1,4} = -\frac{2}{3}$
${}^1\lambda_2 = 3$	$h_{2,1} = -\frac{9}{5}$	$h_{2,2} = -\frac{27}{7}$	$h_{2,3} = -3$	$h_{2,4} = -\frac{9}{2}$
${}^1\lambda_3 = 5$	$h_{3,1} = -\frac{25}{7}$	$h_{3,2} = -\frac{25}{3}$	$h_{3,3} = -\frac{25}{4}$	$h_{3,4} = -10$
${}^1\mu_1 = 0$	$h_{4,1} = 0$	$h_{4,2} = 0$	$h_{4,3} = 0$	$h_{4,4} = 0$
${}^1\mu_2 = 2$	$h_{5,1} = -1$	$h_{5,2} = -2$	$h_{5,3} = -\frac{8}{5}$	$h_{5,4} = -\frac{16}{7}$
${}^1\mu_3 = 4$	$h_{6,1} = -\frac{8}{3}$	$h_{6,2} = -6$	$h_{6,3} = -\frac{32}{7}$	$h_{6,4} = -\frac{64}{9}$

$$\xrightarrow{\mathcal{N}(\mathbb{L}_2)} \mathbf{c}_2 = \begin{bmatrix} -\frac{1}{3} \\ \frac{5}{9} \\ \hline \frac{10}{9} \\ -\frac{14}{9} \\ \hline -\frac{7}{9} \\ 1 \end{bmatrix}$$

- 1 $\mathbb{L}_1$ along ${}^1s$, for ${}^2s = {}^2\lambda_2 = -3$
- 3 $\mathbb{L}_1$ along ${}^2s$ for ${}^1s = \{{}^1\lambda_1, {}^1\lambda_2, {}^1\lambda_3\}$
- Scaled null space $\mathbf{c}_2^\top =$ $\left[\ \mathbf{c}_1^{{}^1\lambda_1} \cdot [\mathbf{c}_1^{{}^2\lambda_2}]_1 \quad \mathbf{c}_1^{{}^1\lambda_2} \cdot [\mathbf{c}_1^{{}^2\lambda_2}]_2 \quad \mathbf{c}_1^{{}^1\lambda_3} \cdot [\mathbf{c}_1^{{}^2\lambda_2}]_3\ \right]^\top$

Taming the curse of dimensionality

2-D case (example cont'd, to get the idea)

Data generated from $\mathbf{H}({}^1s, {}^2s) = \mathbf{H}(s,t) = (s^2 t)/(s-t+1)$ of complexity $(2,1)$

${}^1s \backslash {}^2s$	${}^2\lambda_1 = -1$	${}^2\lambda_2 = -3$	${}^2\mu_1 = -2$	${}^2\mu_2 = -4$
${}^1\lambda_1 = 1$	$h_{1,1} = -\frac{1}{3}$	$h_{1,2} = -\frac{3}{5}$	$h_{1,3} = -\frac{1}{2}$	$h_{1,4} = -\frac{2}{3}$
${}^1\lambda_2 = 3$	$h_{2,1} = -\frac{9}{5}$	$h_{2,2} = -\frac{27}{7}$	$h_{2,3} = -3$	$h_{2,4} = -\frac{9}{2}$
${}^1\lambda_3 = 5$	$h_{3,1} = -\frac{25}{7}$	$h_{3,2} = -\frac{25}{3}$	$h_{3,3} = -\frac{25}{4}$	$h_{3,4} = -10$
${}^1\mu_1 = 0$	$h_{4,1} = 0$	$h_{4,2} = 0$	$h_{4,3} = 0$	$h_{4,4} = 0$
${}^1\mu_2 = 2$	$h_{5,1} = -1$	$h_{5,2} = -2$	$h_{5,3} = -\frac{8}{5}$	$h_{5,4} = -\frac{16}{7}$
${}^1\mu_3 = 4$	$h_{6,1} = -\frac{8}{3}$	$h_{6,2} = -6$	$h_{6,3} = -\frac{32}{7}$	$h_{6,4} = -\frac{64}{9}$

$$\xrightarrow{\mathcal{N}(\mathbb{L}_2)} \mathbf{c}_2 = \begin{bmatrix} -\frac{1}{3} \\ \frac{5}{9} \\ \hline \frac{10}{9} \\ -\frac{14}{9} \\ \hline -\frac{7}{9} \\ 1 \end{bmatrix}$$

- 1 $\mathbb{L}_1$ along ${}^1s$, for ${}^2s = {}^2\lambda_2 = -3$
- 3 $\mathbb{L}_1$ along ${}^2s$ for ${}^1s = \{{}^1\lambda_1, {}^1\lambda_2, {}^1\lambda_3\}$
- Scaled null space $\mathbf{c}_2^\top = \left[\ \mathbf{c}_1^{{}^1\lambda_1} \cdot [\mathbf{c}_1^{{}^2\lambda_2}]_1 \quad \mathbf{c}_1^{{}^1\lambda_2} \cdot [\mathbf{c}_1^{{}^2\lambda_2}]_2 \quad \mathbf{c}_1^{{}^1\lambda_3} \cdot [\mathbf{c}_1^{{}^2\lambda_2}]_3\ \right]^\top$

$$\mathbf{c}_1^{{}^2\lambda_2} = \begin{bmatrix} \frac{5}{9} \\ -\frac{14}{9} \\ 1 \end{bmatrix}$$

Taming the curse of dimensionality

2-D case (example cont'd, to get the idea)

Data generated from $\mathbf{H}({}^1s, {}^2s) = \mathbf{H}(s,t) = (s^2t)/(s-t+1)$ of complexity $(2,1)$

${}^1s \backslash {}^2s$	${}^2\lambda_1 = -1$	${}^2\lambda_2 = -3$	${}^2\mu_1 = -2$	${}^2\mu_2 = -4$
${}^1\lambda_1 = 1$	$h_{1,1} = -\frac{1}{3}$	$h_{1,2} = -\frac{3}{5}$	$h_{1,3} = -\frac{1}{2}$	$h_{1,4} = -\frac{2}{3}$
${}^1\lambda_2 = 3$	$h_{2,1} = -\frac{9}{5}$	$h_{2,2} = -\frac{27}{7}$	$h_{2,3} = -3$	$h_{2,4} = -\frac{9}{2}$
${}^1\lambda_3 = 5$	$h_{3,1} = -\frac{25}{7}$	$h_{3,2} = -\frac{25}{3}$	$h_{3,3} = -\frac{25}{4}$	$h_{3,4} = -10$
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${}^1\mu_2 = 2$	$h_{5,1} = -1$	$h_{5,2} = -2$	$h_{5,3} = -\frac{8}{5}$	$h_{5,4} = -\frac{16}{7}$
${}^1\mu_3 = 4$	$h_{6,1} = -\frac{8}{3}$	$h_{6,2} = -6$	$h_{6,3} = -\frac{32}{7}$	$h_{6,4} = -\frac{64}{9}$

$$\xrightarrow{\mathcal{N}(\mathbb{L}_2)} \mathbf{c}_2 = \begin{bmatrix} -\frac{1}{3} \\ \frac{5}{9} \\ \hline \frac{10}{9} \\ -\frac{14}{9} \\ \hline -\frac{7}{9} \\ 1 \end{bmatrix}$$

- 1 $\mathbb{L}_1$ along ${}^1s$, for ${}^2s = {}^2\lambda_2 = -3$
- 3 $\mathbb{L}_1$ along ${}^2s$ for ${}^1s = \{{}^1\lambda_1, {}^1\lambda_2, {}^1\lambda_3\}$

$$\mathbf{c}_1^{{}^2\lambda_2} = \begin{bmatrix} \frac{5}{9} \\ -\frac{14}{9} \\ 1 \end{bmatrix} \text{ and } \mathbf{c}_1^{{}^1\lambda_1} = \begin{bmatrix} -\frac{3}{5} \\ 1 \end{bmatrix}, \mathbf{c}_1^{{}^1\lambda_2} = \begin{bmatrix} -\frac{5}{7} \\ 1 \end{bmatrix}, \mathbf{c}_1^{{}^1\lambda_3} = \begin{bmatrix} -\frac{7}{9} \\ 1 \end{bmatrix}$$

- Scaled null space $\mathbf{c}_2^\top =$

$$\left[\ \mathbf{c}_1^{{}^1\lambda_1} \cdot [\mathbf{c}_1^{{}^2\lambda_2}]_1 \quad \mathbf{c}_1^{{}^1\lambda_2} \cdot [\mathbf{c}_1^{{}^2\lambda_2}]_2 \quad \mathbf{c}_1^{{}^1\lambda_3} \cdot [\mathbf{c}_1^{{}^2\lambda_2}]_3\ \right]^\top$$

Taming the curse of dimensionality

2-D case (example cont'd, to get the idea)

Data generated from $\mathbf{H}({}^1s, {}^2s) = \mathbf{H}(s,t) = (s^2 t)/(s-t+1)$ of complexity $(2,1)$

${}^1s \backslash {}^2s$	${}^2\lambda_1 = -1$	${}^2\lambda_2 = -3$	${}^2\mu_1 = -2$	${}^2\mu_2 = -4$
${}^1\lambda_1 = 1$	$h_{1,1} = -\frac{1}{3}$	$h_{1,2} = -\frac{3}{5}$	$h_{1,3} = -\frac{1}{2}$	$h_{1,4} = -\frac{2}{3}$
${}^1\lambda_2 = 3$	$h_{2,1} = -\frac{9}{5}$	$h_{2,2} = -\frac{27}{7}$	$h_{2,3} = -3$	$h_{2,4} = -\frac{9}{2}$
${}^1\lambda_3 = 5$	$h_{3,1} = -\frac{25}{7}$	$h_{3,2} = -\frac{25}{3}$	$h_{3,3} = -\frac{25}{4}$	$h_{3,4} = -10$
${}^1\mu_1 = 0$	$h_{4,1} = 0$	$h_{4,2} = 0$	$h_{4,3} = 0$	$h_{4,4} = 0$
${}^1\mu_2 = 2$	$h_{5,1} = -1$	$h_{5,2} = -2$	$h_{5,3} = -\frac{8}{5}$	$h_{5,4} = -\frac{16}{7}$
${}^1\mu_3 = 4$	$h_{6,1} = -\frac{8}{3}$	$h_{6,2} = -6$	$h_{6,3} = -\frac{32}{7}$	$h_{6,4} = -\frac{64}{9}$

$$\xrightarrow{\mathcal{N}(\mathbb{L}_2)} \mathbf{c}_2 = \left[\begin{array}{c} -\frac{1}{3} \\ \frac{5}{9} \\ \hline \frac{10}{9} \\ -\frac{14}{9} \\ \hline -\frac{7}{9} \\ 1 \end{array}\right]$$

- 1 $\mathbb{L}_1$ along ${}^1s$, for ${}^2s = {}^2\lambda_2 = -3$
- 3 $\mathbb{L}_1$ along ${}^2s$ for ${}^1s = \{{}^1\lambda_1, {}^1\lambda_2, {}^1\lambda_3\}$
- Scaled null space $\mathbf{c}_2^\top =$

$$\left[\begin{array}{ccc} \mathbf{c}_1^{{}^1\lambda_1} \cdot [\mathbf{c}_1^{{}^2\lambda_2}]_1 & \mathbf{c}_1^{{}^1\lambda_2} \cdot [\mathbf{c}_1^{{}^2\lambda_2}]_2 & \mathbf{c}_1^{{}^1\lambda_3} \cdot [\mathbf{c}_1^{{}^2\lambda_2}]_3 \end{array}\right]^\top$$

$$\mathbf{c}_1^{{}^2\lambda_2} = \begin{bmatrix} \frac{5}{9} \\ -\frac{14}{9} \\ 1 \end{bmatrix}$$ and $$\mathbf{c}_1^{{}^1\lambda_1} = \begin{bmatrix} -\frac{3}{5} \\ 1 \end{bmatrix}, \mathbf{c}_1^{{}^1\lambda_2} = \begin{bmatrix} -\frac{5}{7} \\ 1 \end{bmatrix}, \mathbf{c}_1^{{}^1\lambda_3} = \begin{bmatrix} -\frac{7}{9} \\ 1 \end{bmatrix}$$

Theorem: 2-D to 1-D

Being given the tableau $\mathbf{tab}_2$ tensor in response of the 2-variables $\mathbf{H}({}^1s, {}^2s)$ function, the null space of the corresponding 2-D Loenwer matrix $\mathbb{L}_2$, is spanned by

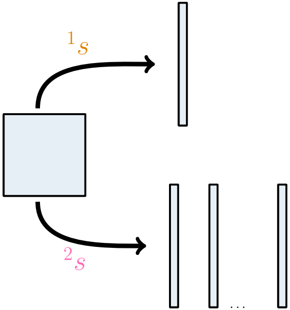
$$\mathcal{N}(\mathbb{L}_2) = \mathbf{vec}\left[\mathbf{c}_1^{{}^2\lambda_1} \cdot \left[\mathbf{c}_1^{{}^1\lambda_{k_1}}\right]_1, \cdots, \mathbf{c}_1^{{}^2\lambda_{k_2}} \cdot \left[\mathbf{c}_1^{{}^1\lambda_{k_1}}\right]_{k_2}\right],$$

where

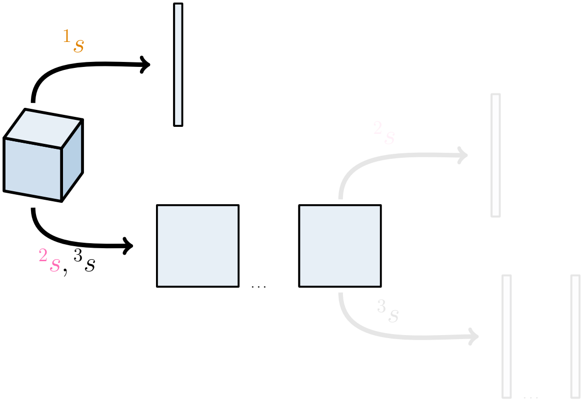
- $\mathbf{c}_1^{{}^1\lambda_{k_1}} = \mathcal{N}(\mathbb{L}_1^{{}^1\lambda_{k_1}})$,
  i.e. the null space of the **1-D Loewner matrix** for frozen ${}^1s = {}^1\lambda_{k_1}$, and
- $\mathbf{c}_1^{{}^2\lambda_{j_2}} = \mathcal{N}(\mathbb{L}_1^{{}^2\lambda_{j_2}})$,
  i.e. the j_1-th null space of the **1-D Loewner matrices** for frozen ${}^2s = \{{}^2\lambda_1, \cdots, {}^2\lambda_{k_2}\}$.

Taming the curse of dimensionality

2-D case

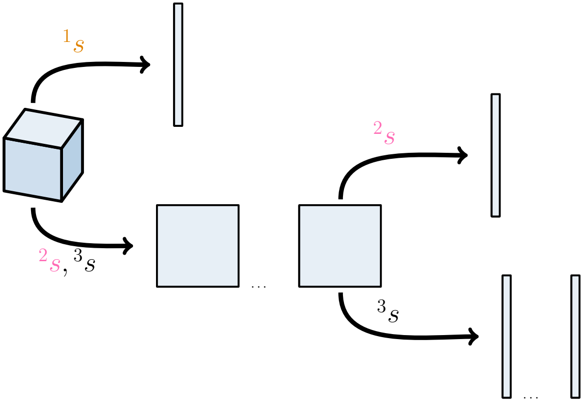


Taming the curse of dimensionality
3-D case



Taming the curse of dimensionality

3-D case



Theorem: n-D to $(n-1)$-D

Being given the tableau $\textbf{tab}_n$ tensor in response of the n-variables $\mathbf{H}({}^1s,\cdots,{}^ns)$ function, the null space of the corresponding n-D Loenwer matrix $\mathbb{L}_n$, is spanned by

$$\mathcal{N}(\mathbb{L}_n) = \mathbf{vec}\left[c_{n-1}^{{}^1\lambda_1} \cdot \left[\mathbf{c}_1^{({}^2\lambda_{k_2},{}^3\lambda_{k_3},\cdots,{}^n\lambda_{k_n})} \right]_1, \cdots, \mathbf{c}_{n-1}^{{}^1\lambda_{k_1}} \cdot \left[\mathbf{c}_1^{({}^2\lambda_{k_2},{}^3\lambda_{k_3},\cdots,{}^n\lambda_{k_n})} \right]_{k_1} \right],$$

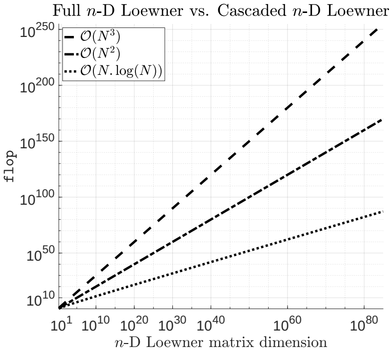
where

- $\mathbf{c}_1^{({}^2\lambda_{k_2},{}^3\lambda_{k_3},\cdots,{}^n\lambda_{k_n})}$ spans $\mathcal{N}(\mathbb{L}_1^{({}^2\lambda_{k_2},{}^3\lambda_{k_3},\cdots,{}^n\lambda_{k_n})})$,
  i.e. the null space of the **1-D Loewner matrix** for frozen $\{{}^2\lambda_{k_2},{}^3\lambda_{k_3},\cdots,{}^n\lambda_{k_n}\}$, and
- $\mathbf{c}_{n-1}^{{}^1\lambda_{j_1}}$ spans $\mathcal{N}(\mathbb{L}_{n-1}^{{}^1\lambda_j})$,
  i.e. the j_1-th null space of the $(n-1)$-**D Loewner matrix** for frozen ${}^1s_{j_1} = \{{}^1\lambda_1,\cdots,{}^1\lambda_{k_1}\}$.

Taming the curse of dimensionality

Null space - `flop` complexity

log-log scale



Null space underlying problem

Let (rows) $Q = q_1 q_2 \dots q_n$ and (columns) $K = k_1 k_2 \dots k_n$

$$\mathbb{L}_n \in \mathbb{C}^{Q \times K}$$

Note that $Q \times K$ matrix SVD `flop` estimation is

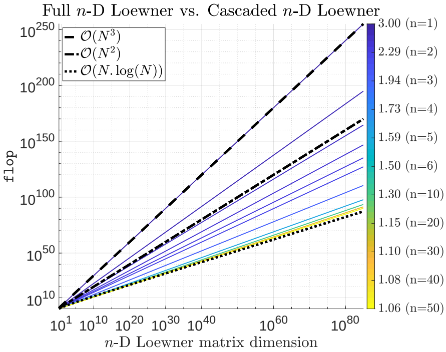
- QK^2 (if $Q > K$)
- N^3 (if $Q = K = N$)

$\Rightarrow$ **The CURSE of dimensionality**

Taming the curse of dimensionality

Null space - `flop` complexity

log-log scale



Theorem: Recursive complexity

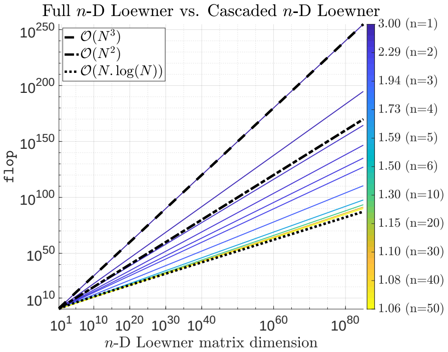
$$\mathtt{flop}_1(n) = \sum_{j=1}^{n} \left(k_j^3 \prod_{l=1}^{j} k_{l-1} \right) \text{ where } k_0 = 1.$$

$\Rightarrow$ **The CURSE of dimensionality is TAMED**

Taming the curse of dimensionality

Null space - `flop` complexity

log-log scale



Corollary: Worst case complexity

k interpolation points per variables.

$$\overline{\texttt{flop}_1} = k^3 \frac{1-k^n}{1-k} = k^3 \frac{1-N}{1-k},$$

which is a (n finite) geometric series of ratio k.

$\Rightarrow$ **The CURSE of dimensionality is TAMED**

$$\begin{array}{llll} \mathcal{O}(N^3) & \rightarrow & \mathcal{O}(N^{2.29}) & \text{for } n=2 \\ & \rightarrow & \mathcal{O}(N^{1.94}) & \text{for } n=3 \\ & \vdots & & \\ & \rightarrow & \mathcal{O}(N^{1.5}) & \text{for } n=6 \\ & \vdots & & \\ & \rightarrow & \mathcal{O}(N^{1.06}) & \text{for } n=50 \end{array}$$

Taming the curse of dimensionality

Null space - memory

With similar importance, the data storage is a key element in the curse of dimensionality. The data (tableau) storage is (complex and double precision)

$$\frac{8}{2^{20}} \prod_l^n q_l + k_l \ \ \texttt{MB} \text{ (example tableau } 2 \cdot [20,6,4,6,8,2] = 2 \cdot [k_1,k_2,k_3,k_4,k_5,k_6] \text{ needs } 45 \ \texttt{MB})$$

Full n-D Loewner

Construction of

$$\mathbb{L}_n \in \mathbb{C}^{N \times N}$$

where $N = k_1 k_2 \cdots k_n$, needs

$$\frac{8}{2^{20}} N^2 \ \ \texttt{MB}$$

Example: $N = 46,080$
Memory: 31.64 `GB`
`flop`: $9.78 \cdot 10^{13}$

Taming the curse of dimensionality

Null space - memory

With similar importance, the data storage is a key element in the curse of dimensionality. The data (tableau) storage is (complex and double precision)

$$\frac{8}{2^{20}}\prod_{l}^{n} q_l + k_l \;\; \texttt{MB} \text{ (example tableau } 2\cdot[20,6,4,6,8,2] = 2\cdot[k_1,k_2,k_3,k_4,k_5,k_6] \text{ needs } 45 \texttt{ MB)}$$

Full n-D Loewner

Construction of

$$\mathbb{L}_n \in \mathbb{C}^{N\times N}$$

where $N = k_1 k_2 \cdots k_n$, needs

$$\frac{8}{2^{20}} N^2 \;\; \texttt{MB}$$

Example: $N = 46{,}080$
Memory: 31.64 `GB`
`flop`: $9.78\cdot 10^{13}$

Cascaded n-D Loewner

Construction of

$$\mathbb{L}_1 \in \mathbb{C}^{\overline{k}\times\overline{k}}$$

where $\overline{k} = \max_j k_j$, needs

$$\frac{8}{2^{20}} \overline{k}^2 \;\; \texttt{MB}$$

Example: $\overline{k} = 20$
Memory: 6.25 `KB`
`flop`: $8.13\cdot 10^{5}$

Taming the curse of dimensionality

Numerical examples, 20-D example

$$\mathbf{H}({}^1s, {}^2s, \dots, {}^{20}s) =$$

$$\frac{3 \cdot {}^1s^3 + 4 \cdot {}^8s + {}^{12}s + {}^{13}s \cdot {}^{14}s + {}^{15}s}{{}^1s + {}^2s^2 \cdot {}^3s + {}^4s + {}^5s + {}^6s + {}^7s \cdot {}^8s + {}^9s \cdot {}^{10}s \cdot {}^{11}s + {}^{13}s + {}^{13}s^3 \cdot \pi + {}^{17}s + {}^{18}s \cdot {}^{19}s - {}^{20}s}$$

Statistics

- 20-D tensor of dimension (≥ 48 TB in real double precision)
- Complexity: $(3, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 3, 1, 1, 1, 1)$
- n-D Loewner matrix $6,291,456^2 \rightarrow 288$ TB of storage in real double precision
- Full SVD: $2.49 \cdot 10^{20}$ `flop`
  Recursive SVD: $5.43 \cdot 10^{7}$ `flop`
- error $\approx 10^{-11}$

Taming the curse of dimensionality

Numerical examples, 20-D example

$$\mathbf{H}({}^{1}s, {}^{2}s, \ldots, {}^{20}s) =$$

$$\frac{3 \cdot {}^{1}s^{3} + 4 \cdot {}^{8}s + {}^{12}s + {}^{13}s \cdot {}^{14}s + {}^{15}s}{{}^{1}s + {}^{2}s^{2} \cdot {}^{3}s + {}^{4}s + {}^{5}s + {}^{6}s + {}^{7}s \cdot {}^{8}s + {}^{9}s \cdot {}^{10}s \cdot {}^{11}s + {}^{13}s + {}^{13}s^{3} \cdot \pi + {}^{17}s + {}^{18}s \cdot {}^{19}s - {}^{20}s}$$

Statistics

- 20-D tensor of dimension (≥ 48 TB in real double precision)
- Complexity: $(3, 3, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$
- n-D Loewner matrix $6,291,456^2 \rightarrow 288$ TB of storage in real double precision
- Full SVD: $2.49 \cdot 10^{20}$ `flop`
  Recursive SVD: $5.43 \cdot 10^{7}$ `flop` $\rightarrow 5.03 \cdot 10^{7}$ `flop`
- error $\approx 10^{-11}$

Taming the curse of dimensionality

Numerical examples (from 2 to 20 variables)

#4 Rational function

$$s_4{}^3 + \frac{s_1\, s_3}{s_3{}^2 + s_1 + s_2 + 1}$$

#5 Rational function

$$\frac{s_3{}^2 + s_1\, s_3\, s_5{}^3}{s_1{}^3 + s_4 + s_2\, s_3}$$

#6 Rational function

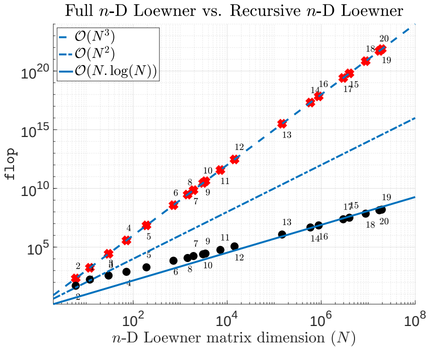
$$\frac{-\sqrt{2}\, s_6{}^2 + s_1 + s_3}{s_1{}^2 + s_4{}^3 + s_5{}^2 + s_6 + s_2\, s_3}$$

#7 Rational function

$$\frac{s_3\, s_2{}^3 + 1}{s_3\, s_2{}^2 + s_4{}^2 + s_6{}^3 + s_1 + s_5 + s_7}$$

#19 Rational function

$$\frac{3\, s_1{}^3 + s_{18}{}^2 + 4\, s_8 + s_{12} + s_{15} + s_{13}\, s_{14}}{s_3\, s_2{}^2 + \pi\, s_{16}{}^3 + s_{17}{}^2 + s_1 + s_4 + s_5 + s_6 + s_{13} + s_{19} + s_7\, s_8 + s_9\, s_{10}\, s_{11}}$$



Taming the curse of dimensionality

Numerical examples (rational and irrational)

#16 Arc-tangent function

$$\frac{\operatorname{atan}(x_1)+\operatorname{atan}(x_2)+\operatorname{atan}(x_3)+\operatorname{atan}(x_4)}{{x_1}^2\,{x_2}^2-{x_1}^2-{x_2}^2+1}$$

#17 Exponential function

$$\frac{e^{x_1\,x_2\,x_3\,x_4}}{{x_1}^2+{x_2}^2-x_3\,x_4+3}$$

#18 Sinc function

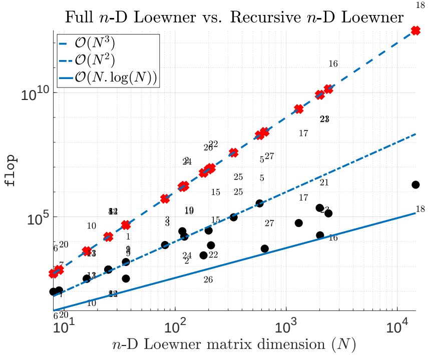
$$\frac{10\,\sin(x_1)\,\sin(x_2)\,\sin(x_3)\,\sin(x_4)}{x_1\,x_2\,x_3\,x_4}$$

#19 Sinc function

$$\frac{10\,\sin(x_1)\,\sin(x_2)}{x_1\,x_2}$$

#20 Polynomial function

$${x_1}^2+x_1\,x_2+{x_2}^2-x_2+1$$



Content

Variable decoupling

Given data $\mathbf{tab}_n$, the (recursive) theorem achieves variables decoupling, and the null space can be equivalently written as:

$$\mathbf{c}_n = \underbrace{\mathbf{c}^{{}^n s}}_{\mathbf{Bary}({}^n s)} \odot \underbrace{\left(\mathbf{c}^{{}^{n-1} s} \otimes \mathbf{1}_{k_n}\right)}_{\mathbf{Bary}({}^{n-1} s)} \odot \underbrace{\left(\mathbf{c}^{{}^{n-2} s} \otimes \mathbf{1}_{k_n k_{n-1}}\right)}_{\mathbf{Bary}({}^{n-2} s)} \odot \cdots \odot \underbrace{\left(\mathbf{c}^{{}^1 s} \otimes \mathbf{1}_{k_n \ldots k_2}\right)}_{\mathbf{Bary}({}^1 s)}.$$

where $\mathbf{c}^{{}^j s}$ denotes the vectorized barycentric coefficients related to the j-th variable.

Variables decoupling, KST and KANs

Loewner, Kolmogorov Superposition Theorem and Hilbert's 13th problem

Variable decoupling

Given data $\mathbf{tab}_n$, the latter achieves variables decoupling, and the null space can be equivalently written as:

$$\mathbf{c}_n = \underbrace{\mathbf{c}^{{}^n s}}_{\mathbf{Bary}({}^n s)} \odot \underbrace{(\mathbf{c}^{{}^{n-1} s} \otimes \mathbf{1}_{k_n})}_{\mathbf{Bary}({}^{n-1} s)} \odot \underbrace{(\mathbf{c}^{{}^{n-2} s} \otimes \mathbf{1}_{k_n k_{n-1}})}_{\mathbf{Bary}({}^{n-2} s)} \odot \cdots \odot \underbrace{(\mathbf{c}^{{}^1 s} \otimes \mathbf{1}_{k_n \ldots k_2})}_{\mathbf{Bary}({}^1 s)}.$$

where $\mathbf{c}^{{}^l s}$ denotes the vectorized barycentric coefficients related to the l-th variable.

"Kolmogorov proved that every continuous function of several variables can be represented as a superposition of continuous functions of one variable and the operation of addition (1957). Thus, it is as if there are no functions of several variables at all. Seriously speaking, Kolmogorov's theorem is a brilliant example of his mastery. In particular, the theorem shows that Hilbert's conjecture (to it's 13th problem) is wrong."

A.G. Vitushkin, *"On Hilbert's thirteenth problem and related questions"*, Russian Math. Surveys 59:1, pp. 11–25.

Variables decoupling, KST and KANs

KANs via Loewner with rational activation functions ($\mathbf{H} = {}^1s \cdot {}^2s$)

$$
\begin{aligned}
{}^1\lambda_{j_1} &= \begin{pmatrix} -1 & 1 \end{pmatrix} \\
{}^2\lambda_{j_2} &= \begin{pmatrix} -1 & 1 \end{pmatrix}
\end{aligned}
$$

$$
\begin{pmatrix}
\mathbf{c} & \mathbf{w} & \mathbf{c}\cdot\mathbf{w} & \mathbf{Lag} \\
1.0 & 1.0 & 1.0 & \frac{1}{({}^1s+1.0)\,({}^2s+1.0)} \\
-1.0 & -1.0 & 1.0 & \frac{1}{({}^1s+1.0)\,({}^2s-1.0)} \\
-1.0 & -1.0 & 1.0 & \frac{1}{({}^1s-1.0)\,({}^2s+1.0)} \\
1.0 & 1.0 & 1.0 & \frac{1}{({}^1s-1.0)\,({}^2s-1.0)}
\end{pmatrix}
$$

$$
\begin{aligned}
\mathbf{c}^{{}^2s} &= \textbf{vec}\begin{pmatrix} -1.0 & -1.0 \\ 1.0 & 1.0 \end{pmatrix} \\
\mathbf{c}^{{}^1s} &= \begin{pmatrix} -1.0 \\ 1.0 \end{pmatrix}
\end{aligned}
$$

$$
\mathbf{c}_2 = \mathbf{c}^{{}^2s} \odot (\mathbf{c}^{{}^1s} \otimes \mathbf{1}_{k_2})
$$

KAN with rational activation functions

$$
\mathbf{D} = \begin{pmatrix}
\mathbf{c}^{{}^1s}\cdot\mathbf{Lag}({}^1s) & \mathbf{c}^{{}^2s}\cdot\mathbf{Lag}({}^2s) \\
-\frac{1.0}{{}^1s+1.0} & -\frac{1.0}{{}^2s+1.0} \\
-\frac{1.0}{{}^1s+1.0} & \frac{1}{{}^2s-1.0} \\
\frac{1}{{}^1s-1.0} & -\frac{1.0}{{}^2s+1.0} \\
\frac{1}{{}^1s-1.0} & \frac{1}{{}^2s-1.0}
\end{pmatrix}
$$

Equivalent denominator and numerator read:

$$
\sum_{i\text{-th row}} \prod_{j\text{-th col}} [\mathbf{D}]_{i,j} \quad \text{and} \quad \sum_{i\text{-th row}} \mathbf{w}\cdot \prod_{j\text{-th col}} [\mathbf{D}]_{i,j}
$$

Variables decoupling, KST and KANs

KANs via Loewner with rational activation functions ($\mathbf{H} = {}^1s \cdot {}^2s$)

$$
\begin{aligned}
{}^1\lambda_{j_1} &= \begin{pmatrix} -1 & 1 \end{pmatrix} \\
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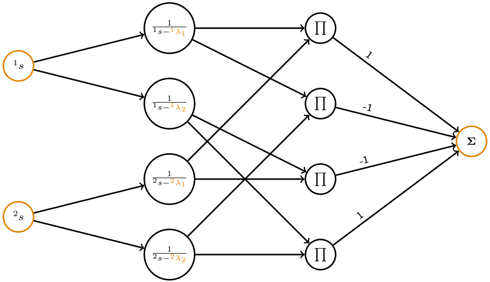
\end{aligned}
$$

$$
\begin{pmatrix}
\mathbf{c} & \mathbf{w} & \mathbf{c}\cdot\mathbf{w} & \mathbf{Lag} \\
1.0 & 1.0 & 1.0 & \frac{1}{({}^1s+1.0)\,({}^2s+1.0)} \\
-1.0 & -1.0 & 1.0 & \frac{1}{({}^1s+1.0)\,({}^2s-1.0)} \\
-1.0 & -1.0 & 1.0 & \frac{1}{({}^1s-1.0)\,({}^2s+1.0)} \\
1.0 & 1.0 & 1.0 & \frac{1}{({}^1s-1.0)\,({}^2s-1.0)}
\end{pmatrix}
$$

$$
\begin{aligned}
\mathbf{c}^{{}^2s} &= \mathbf{vec}\begin{pmatrix} -1.0 & -1.0 \\ 1.0 & 1.0 \end{pmatrix} \\
\mathbf{c}^{{}^1s} &= \begin{pmatrix} -1.0 \\ 1.0 \end{pmatrix}
\end{aligned}
$$

$$\mathbf{c}_2 = \mathbf{c}^{{}^2s} \odot (\mathbf{c}^{{}^1s} \otimes \mathbf{1}_{k_2})$$

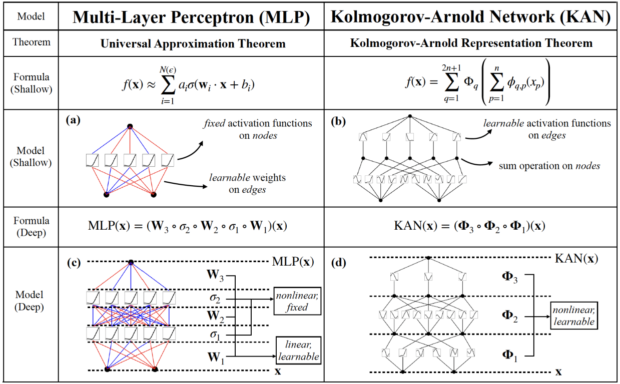
KAN with rational activation functions



A.G. Vitushkin, *"On Hilbert's thirteenth problem and related questions"*, Russian Math. Surveys 59:1, pp. 11–25.

KANs features

- **Inspired by the Kolmogorov-Arnold representation theorem**
- The model output is a composition of **sums** and **learnable activation functions** (e.g. splines)
- Alternate to Multi-Layer Perceptrons (MLP), having fixed activation functions (e.g. ReLU), **inspired by the universal approximation theorem**



Comparison between MLP and KAN (figure from *Z. Liu et al.*)

Z. Liu et al., *"KAN: Kolmogorov–Arnold Networks"*, `https://arxiv.org/abs/2404.19756`.

Building the Kolmogorov-Arnold model as follows

$$F(^1s, ^2s, \cdots, ^ns) = \sum_{k=1}^{2m+1} \Phi_k \left(\sum_{j=1}^{m} f_{kj}(^js) \right)$$

$f_{kj} : [0,1] \mapsto \mathbb{R}$ and $\Phi_k : \mathbb{R} \mapsto \mathbb{R}$ are continuous functions.

The relation is approximated by $k = 1, \cdots, d = 2m+1$ as

$$\hat{F}(^1s, ^2s, \cdots, ^ns) = \sum_{k=1}^{d} \Phi_k \underbrace{\left(\sum_{j=1}^{m} f_{kj}(^js_i) \right)}_{\theta_{ik}}$$

where θ_{ik} denotes the k-th component of θ_i vector (interpreted as a hidden variable between two layers), which describes **splines**

M. Poluektov and A. Polar, *"Construction of the Kolmogorov-Arnold representation using the Newton-Kaczmarz method"*, `https://arxiv.org/abs/2305.08194`.

Variables decoupling, KST and KANs

KANs with splines

Building the Kolmogorov-Arnold model as follows

$$F({}^1s, {}^2s, \cdots, {}^ns) = \sum_{k=1}^{2m+1} \Phi_k \left(\sum_{j=1}^{m} f_{kj}({}^js) \right)$$

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Variables decoupling, KST and KANs

KANs (via Loewner) with rational activation functions

$$\mathbf{H} = {}^1s \cdot {}^2s$$

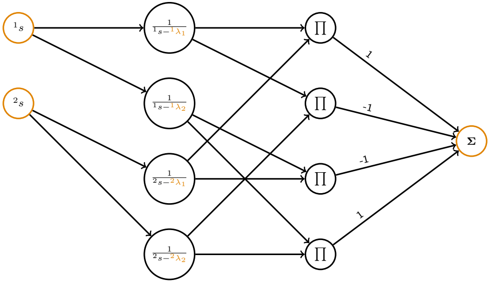
$$\begin{array}{rcl} {}^1\lambda_{j_1} & = & \begin{pmatrix} -1 & 1 \end{pmatrix} \\ {}^2\lambda_{j_2} & = & \begin{pmatrix} -1 & 1 \end{pmatrix} \end{array}$$

$$\mathbf{D} = \begin{pmatrix} \mathbf{c}^{{}^1s} \cdot \mathbf{Lag}({}^1s) & \mathbf{c}^{{}^2s} \cdot \mathbf{Lag}({}^2s) \\ -\frac{1.0}{{}^1s+1.0} & -\frac{1.0}{{}^2s+1.0} \\ -\frac{1.0}{{}^1s+1.0} & \frac{1}{{}^2s-1.0} \\ \frac{1}{{}^1s-1.0} & -\frac{1.0}{{}^2s+1.0} \\ \frac{1}{{}^1s-1.0} & \frac{1}{{}^2s-1.0} \end{pmatrix}$$

Equivalent denominator and numerator read:

$$\sum_{i\text{-th row}} \prod_{j\text{-th col}} [\mathbf{D}]_{i,j} \text{ and } \sum_{i\text{-th row}} \mathbf{w} \cdot \prod_{j\text{-th col}} [\mathbf{D}]_{i,j}$$

Equivalent KAN-like with rational activation functions (just $\mathbf{D}$)



Content

Comparisons

Some competitors

KAN (P/P 2025)

- Kolmogorov Arnold Network

Rat. app (B/G 2025)

- p-AAA
- Dense algebra

MLP (TensorFlow 2025)

- Multi Layer Perceptron
- Keras is the high-level API of TensorFlow (by Google) `https://www.tensorflow.org/guide/keras?hl=en`
- Dense connected network / ReLU activation / ADAM optim. / 1000 iterations / random init.

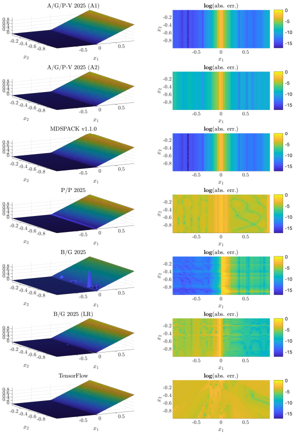
`TensorFlow interface` (Python code)

```python
import numpy as np
import math
import matplotlib.pyplot as plt
import tensorflow as tf
from keras.models import Sequential
from matplotlib import cm

# Data
def H(x):
    #y = pow(x[:,0],2)*x[:,1]
    #y = 1/2*(x[:,0] + np.abs(x[:,0])) + 1/10*x[:,1] #1
    #y = x[:,0]*x[:,1] #3
    #y = np.exp(x[:,0]*x[:,1])/( (pow(x[:,0],2)-1.44)*(pow(x[:,1],2)-1.44) ) #6
    #y = np.tanh(4*(x[:,0]-x[:,1])) #8
    #y = pow(np.abs((x[:,0]-x[:,1])),3) #10
    y = (pow(x[:,0],2) + pow(x[:,1],2) + x[:,0] - x[:,1] + 1) / (pow(x[:,0],3)  + pow(x[:,1],2) + 4) #15
    return np.transpose(np.array([y]))

n   = 2
N1  = 40
N2  = 40
x1  = np.linspace(-1, 1, N1)
x2  = np.linspace(-1, 1, N2)

# IP
N   = N1*N2
tab = np.zeros((N,1))
```

`https://arxiv.org/abs/2506.04791`
`https://github.com/cpoussot/mLF`

Comparisons

Irrational functions (example #1)



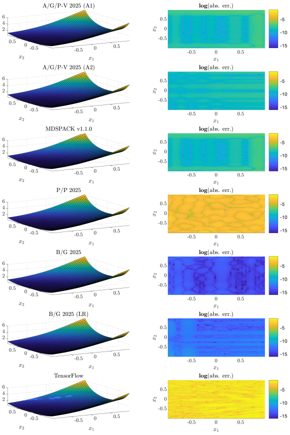
#	Alg.	Parameters	Dim.	CPU [s]	RMSE	min err.
1	A/G/P-V 2025 (A1)	1e-11,3	144	**0.0171**	0.000699	$\mathbf{1.39e-17}$
	A/G/P-V 2025 (A2)	1e-15,3	160	0.0823	**0.000389**	4.49e-13
	MDSPACK v1.1.0	1e-11,1e-06	144	0.0623	0.000699	2.43e-17
	P/P 2025	1,0.95,50,0.01,4,6,9	**130**	0.282	0.0017	3.08e-07
	B/G 2025	0.001,20	612	0.353	0.0288	3.75e-16
	B/G 2025 (LR)	0.001,20,4	480	0.678	0.00147	7.82e-12
	TensorFlow		257	14.6	0.00074	6.31e-08

$$\mathrm{ReLU}({}^{1}s) + \frac{1}{100}\,{}^{2}s$$

- Reference: Personal communication, [none]
- Domain: $\mathbb{R}$
- Tensor size: 12.5 **KB** (40^2 points)
- Bounds: $\begin{pmatrix} -1 & 1 \end{pmatrix} \times \begin{pmatrix} -1 & -10^{-10} \end{pmatrix}$

Comparisons

Irrational functions (example #2)



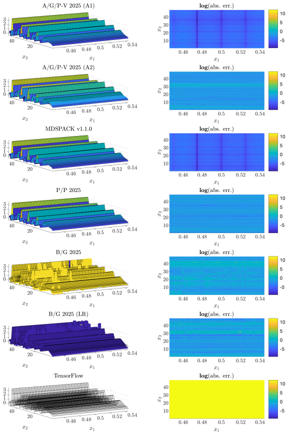
#	Alg.	Parameters	Dim.	CPU [s]	RMSE	min err.	max e
2	A/G/P-V 2025 (A1)	1e-06,1	**168**	0.0235	3.36e-08	4.2e-13	1.7e-0
	A/G/P-V 2025 (A2)	1e-15,1	196	0.0888	4.48e-09	2.58e-12	1.62e-
	MDSPACK v1.1.0	1e-06,0.0001	168	**0.0161**	3.36e-08	2.09e-12	1.7e-0
	P/P 2025	1,1,50,0.01,6,6,13	238	0.591	0.000287	1.84e-07	0.001
	B/G 2025	1e-09,20	396	0.0665	$\mathbf{4.88e-14}$	**0**	**4.32**
	B/G 2025 (LR)	1e-09,20,3	480	0.827	1.29e-12	2.22e-16	1.38e-
	TensorFlow		257	14.6	0.00937	9.75e-05	0.036

$$\mathbf{exp}(\mathbf{sin}({}^1s) + {}^2s^2)$$

- Reference: Personal communication, [none]
- Domain: $\mathbb{R}$
- Tensor size: 12.5 **KB** (40^2 points)
- Bounds: $\begin{pmatrix} -1 & 1 \end{pmatrix} \times \begin{pmatrix} -1 & 1 \end{pmatrix}$

Comparisons

Irrational functions (example #34)



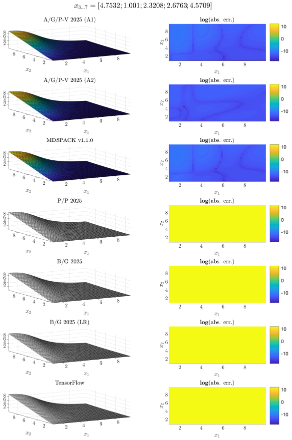
#	Alg.	Parameters	Dim.	CPU [s]	RMSE	min err.
34	A/G/P-V 2025 (A1)	1e-10,2	2.32e+03	1.5	4.86e-05	**2.74e −**
	A/G/P-V 2025 (A2)	1e-15,2	864	**0.654**	0.137	1.51e-06
	MDSPACK v1.1.0	1e-10,1e-09	2.3e+03	1.58	**3.21e − 05**	1.87e-08
	P/P 2025	1,0.95,50,0.01,10,12,21	**676**	79	0.0254	1.36e-05
	B/G 2025	0.001,20	1.22e+03	85.6	2.53	0.000305
	B/G 2025 (LR)	1e-06,20,3	1.36e+03	16.9	4.74	1.23e-05
	TensorFlow	NaN	NaN	NaN	NaN	NaN

$$\mathrm{Re}(\zeta({}^1s + \imath\,{}^2s))$$

- Riemann ζ function (real part), [none]
- Domain: $\mathbb{R}$
- Tensor size: 1.22 **MB** (400^2 points)
- Bounds: $\begin{pmatrix} \frac{9}{20} & \frac{11}{20} \end{pmatrix} \times \begin{pmatrix} 1 & 50 \end{pmatrix}$

Comparisons

Irrational functions (example #43)



#	Alg.	Parameters	Dim.	CPU [s]	RMSE	min err.
43	A/G/P-V 2025 (A1)	0.0001,1	**1.73e + 04**	**5.58**	1.41e-12	1.76e-16
	A/G/P-V 2025 (A2)	1e-15,1	1.73e+04	12.5	**2.39e − 13**	**1.21e − 17**
	MDSPACK v1.1.0	0.0001,1e-06	1.73e+04	5.72	1.4e-12	1.49e-16
	P/P 2025	NaN	NaN	NaN	NaN	NaN
	B/G 2025	NaN	NaN	NaN	NaN	NaN
	B/G 2025 (LR)	NaN	NaN	NaN	NaN	NaN
	TensorFlow	NaN	NaN	NaN	NaN	NaN

$$\frac{{}^3s\ {}^2s^3 + 1}{{}^1s^4 + {}^2s^2\ {}^3s + {}^4s^2 + {}^5s + {}^6s^3 + {}^7s}$$

- Reference: Personal communication, [none]
- Domain: $\mathbb{R}$
- Tensor size: $\mathbb{R}$
- Tensor size: 76.3 **MB** (10^7 points)
- Bounds: $\begin{pmatrix} 1 & 10 \end{pmatrix}^7$

Content

Conclusion

Take home message

Main contributions

From any n-th order multi-variate transfer function / data tensor

- Construct a transfer function in barycentric form
- **Construct a realization with controlled complexity**
- **Tame the computational complexity**
  $\mathcal{O}(N^3) \rightarrow\approx \mathcal{O}(N^{2.29,1.94,\cdots})$
- Two algorithms (direct & iterative)
- Connection with Kolmogorov theorem

Side effects

[Theo.] Tame COD in KST

[Sci. con.] Tensor rank approximation

[Sci. con.] Achieve multi-linearization of NEVP

[Sci. con.] Exact (Loewner) matrix null space computation

[Dyn. sys.] Multi-variate / parametric realization

Collaboration with
A.C. Antoulas [Rice Univ.]
I.V. Goșea [MPI]

`https://arxiv.org/abs/2405.00495`
`https://github.com/cpoussot/mLF`

Conclusion

Big picture