# Linear dynamical system identification

... basic elements and Labs guidelines

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February 20, 2026

# Introduction

## Course plan, overview and questions addressed

### L1: Overview, signals construction, pre-treatment and non-parametric analysis

- Realization, transfer functions, ...
- Construct an experimental plan and signals.
- How to analyze signals, and derive some properties?

### L2: Data-driven model construction in the time- and frequency-domain

- Construct a linear model from time- or frequency-domain data.
- How much is it valid? How to validate, discuss, amend it?

### L3: L2 cont'd & Labs guidelines

- Illustration in practice
- Experimental setup & numerical tools presentation.
- Methodology for the lab.

# Introduction

## Today

**L2: Data-driven model construction in the time-domain & frequency**

- Construct a linear model from time- or frequency-domain data.
- How much is it valid? How to validate, discuss, amend it?

### Notions treated

- Some linear algebra reminder
- Time-domain $\rightarrow$ `n4sid` (by Matlab)
  N4SID - Numerical algorithms for Subspace State-Space System IDentification
  N4SID and Hankel
- Frequency-domain $\rightarrow$ `insapack.loewner_tgn` (by pedagogical team)
  LF - Loewner Framework
  LF for data interpolation
  LF for minimal realization
  LF for identification

# Introduction

## Some history & references (linear only!)

### Time-domain

- Deterministic SIMs [Ho and Kalman, 1966]
- Multivariable Output-Error State Space [Verhaegen and Dewilde, 1992]
- N4SID [Van Overschee and De Moor, 1994] [Viberg, 1994]
- Pencil method [Antoulas and Ionita, 2016]

### Frequency-domain & Complex-domain

- N4SID applicable for frequency-domain [McKelvey, Akcay and Ljung, 1996]
- (p)AAA [Nakatsukasa, Sète and Trefethen, 2018] [Balicki and Gugercin, 2025]
- Vector Fitting (...)
- LF [Mayo and Antoulas, 2007] [Antoulas, Ionita, Lefteriu, 2016]
- mLF or (p)LF [Antoulas, Gosea, P-V., 2025] [P-V., Gosea, Vuillemin, Antoulas, 2025]

### Eigenvalues

Given $n \times n$ square matrix $\mathbf{A}, \mathbf{E}$ of real or complex numbers, an eigenvalue $\lambda$ and its associated (non-zero) generalized eigenvector $\mathbf{v}$ are a pair obeying the relation

$$(\mathbf{A} - \lambda I)\mathbf{v} = 0 \text{ or } (\mathbf{A} - \lambda \mathbf{E})\mathbf{v} = 0$$

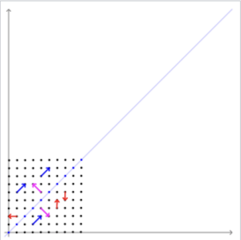
$$\mathbf{A}\mathbf{v} = \lambda \mathbf{v} \text{ or } \mathbf{A}\mathbf{v} = \lambda \mathbf{E}\mathbf{v}$$

The eigenvalues of $A$ are values of $\lambda$ that satisfy the equation

$$det(\mathbf{A} - \lambda I) = 0 \text{ or } det(\mathbf{A} - \lambda \mathbf{E}) = 0$$

Example (wikipedia)

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \lambda = \{1, 3\}, \mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



The transformation matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ preserves the direction of magenta vectors parallel to $\mathbf{v}_{\lambda=1} = [1\ -1]^{\mathrm{T}}$ and blue vectors parallel to $\mathbf{v}_{\lambda=3} = [1\ 1]^{\mathrm{T}}$. The red vectors are not parallel to either eigenvector, so, their directions are changed by the transformation. The lengths of the magenta vectors are unchanged after the transformation (due to their eigenvalue of 1), while blue vectors are three times the length of the original (due to their eigenvalue of 3). See also: An extended version, showing all four quadrants.

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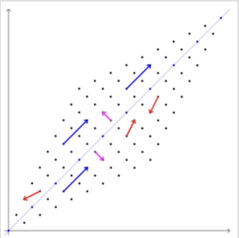

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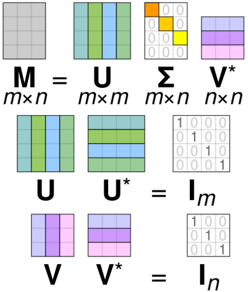
## Singular Value Decomposition

The SVD is a factorization of a real or complex matrix into a rotation, followed by a rescaling followed by another rotation. It generalizes the eigen-decomposition of a square normal matrix with an orthonormal eigen-basis to any $m \times n$ matrix.

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^H$$

- $\mathbf{U} \in \mathbb{C}^{m\times m}$ unitary matrix,
- $\mathbf{\Sigma} \in \mathbb{C}^{m\times n}$ non-negative real numbers on the diagonal
- $\mathbf{V} \in \mathbb{C}^{n\times n}$ unitary matrix,



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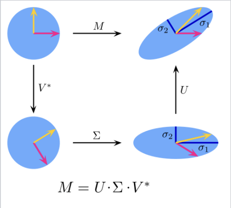



Illustration of the singular value decomposition $\mathbf{U\Sigma V^*}$ of a real $2 \times 2$ matrix $\mathbf{M}$.
**Top:** The action of $\mathbf{M}$, indicated by its effect on the unit disc $D$ and the two canonical unit vectors $e_1$ and $e_2$.
**Left:** The action of $\mathbf{V^*}$, a rotation, on $D$, $e_1$, and $e_2$.
**Bottom:** The action of $\mathbf{\Sigma}$, a scaling by the singular values $\sigma_1$ horizontally and $\sigma_2$ vertically.
**Right:** The action of $\mathbf{U}$, another rotation.

# Introduction

## Linear algebra reminders (SVD applications)

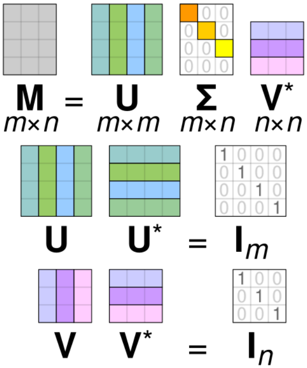
- Pseudoinverse (solving linear least square $\mathbf{A}\mathbf{x} = \mathbf{v}$)

$$\mathbf{A}^{\dagger} = \mathbf{V}\mathbf{\Sigma}^{\dagger}\mathbf{U}^{H}$$

- Null space ($\mathbf{A}\mathbf{x} = 0$) is spanned by the last row of $\mathbf{V}$ i.e. $\mathbf{V}_n$
- Rank is the number of non-zero singular values $\mathbf{\Sigma}$.
- Low rank app.: Eckart-Young theorem

$$\mathbf{A}_k = \sum_{j=1}^{k} \sigma_j \mathbf{u}_j \mathbf{v}_j^H$$

- Image compression
- Separable model, tensor decomposition
- and way more...



# Content

# Time-domain identification (subspace method)

## Basic state-space concepts reminder

### Reachability & Observability

$$\mathcal{R}_n(A,B) = \begin{bmatrix} B & AB & A^2B & \cdots & A^nB \end{bmatrix}$$

$$\mathcal{O}_n(A,C) = \begin{bmatrix} C^\top & A^\top C^\top & (A^\top)^2C^\top & \cdots & (A^\top)^nC^\top \end{bmatrix}^\top$$

### Hankel matrix and Markov parameters

$$\mathcal{H}_n = \mathcal{O}_n\mathcal{R}_n = \begin{bmatrix} H_1 & H_2 & \cdots & H_n \\ H_2 & H_3 & \cdots & H_{n+1} \\ \vdots & & \ddots & \vdots \\ H_n & H_{n+1} & \cdots & H_{2n-1} \end{bmatrix}$$

where

$$H_0 = D \ , \ H_k = CA^{k-1}B \ (k \geq 1)$$

# Time-domain identification (subspace method)

## Identifying $A$, $C$ (SISO)

### Singular Value Decomposition & Rank

Compute the SVD

$$
\begin{aligned}
\mathcal{H}_n &= USV^\top \\
&= \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} S_1 & \\ & S_2 \end{bmatrix} \begin{bmatrix} V_1^\top \\ V_2^\top \end{bmatrix}
\end{aligned}
$$

then, estimate the rank as

$$\dim(S_1) = \textbf{rank}\ (\mathcal{H}_n) = r$$

where $\mathbf{y} = [y_1, \cdots, y_{2N-1}]$ is the impulse response

$$\mathcal{H}_n = \begin{bmatrix} y_1 & y_2 & \cdots & y_N \\ y_2 & y_3 & \cdots & y_{N+1} \\ \vdots & & \ddots & \vdots \\ y_N & y_{N+1} & \cdots & y_{2N-1} \end{bmatrix}$$

# Time-domain identification (subspace method)

## Identifying $A$, $C$ (SISO)

### Singular Value Decomposition & Rank

Compute the SVD

$$\mathcal{H}_n = USV^\top = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} S_1 & \\ & S_2 \end{bmatrix} \begin{bmatrix} V_1^\top \\ V_2^\top \end{bmatrix}$$

$$\dim(S_1) = \mathbf{rank}\ (\mathcal{H}_n) = r$$

Now define ($U_1 \in \mathbb{R}^{N \times r}$)

$$\hat{\mathcal{O}}_n = U_1 S_1^{1/2}$$

and set

$$\begin{array}{rcl} J_1 & = & \begin{bmatrix} I_{N-1} & \mathbf{0}_{N-1,1} \end{bmatrix} \\ J_2 & = & \begin{bmatrix} \mathbf{0}_{N-1,1} & I_{N-1} \end{bmatrix} \\ J_3 & = & \begin{bmatrix} 1 & \mathbf{0}_{1,N-1} \end{bmatrix} \end{array}$$

Thus,

$$\hat{A} = (J_1 U_1)^{-1}(J_2 U_1) \text{ and } \hat{C} = J_3 U_1$$

# Time-domain identification (subspace method)

## Identifying $A$, $C$ (SISO)

### Indeed

Recall, with

$$\begin{array}{rcl} J_1 & = & \begin{bmatrix} I_{N-1} & \mathbf{0}_{N-1,1} \end{bmatrix} \in \mathbb{R}^{(N-1)\times N} \\ J_2 & = & \begin{bmatrix} \mathbf{0}_{N-1,1} & I_{N-1} \end{bmatrix} \in \mathbb{R}^{(N-1)\times N} \\ J_3 & = & \begin{bmatrix} 1 & \mathbf{0}_{1,N-1} \end{bmatrix} \in \mathbb{R}^{1\times N} \end{array}$$

we have

$$J_1 \begin{bmatrix} C \\ CA \\ \vdots \\ C^{N-1} \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ C^{N-2} \end{bmatrix} \text{ and } J_2 \begin{bmatrix} C \\ CA \\ \vdots \\ C^{N-1} \end{bmatrix} = \begin{bmatrix} CA \\ CA^2 \\ \vdots \\ C^{N-1} \end{bmatrix}$$

and thus

$$J_1 \hat{\mathcal{O}}_n \hat{A} = J_2 \hat{\mathcal{O}}_n \text{ or equivalently } J_1 U_1 \hat{A} = J_2 U_1$$

# Time-domain identification (subspace method)

## Identifying $B$, $D$ (SISO)

### Estimate input matrices

$$\hat{B}, \hat{D} := \arg\min_{B,D} \sum_{k=1}^{N} ||\mathbf{y}(k) - \hat{\mathbf{y}}(k)||_F^2$$

### In practice

$$\begin{array}{rcl}
y_0 & = & CB + D & = & \begin{bmatrix} C & 1 \end{bmatrix} \begin{bmatrix} B \\ D \end{bmatrix} \\
y_1 & = & CAB + D & = & \begin{bmatrix} CA & 1 \end{bmatrix} \begin{bmatrix} B \\ D \end{bmatrix} \\
 & \vdots & & & \\
y_N & = & CA^N B + D & = & \begin{bmatrix} CA^N & 1 \end{bmatrix} \begin{bmatrix} B \\ D \end{bmatrix}
\end{array}$$

# Time-domain identification (subspace method)

## Identifying $B$, $D$ (SISO)

### Estimate input matrices

$$\hat{B}, \hat{D} := \arg\min_{B,D} \sum_{k=1}^{N} ||\mathbf{y}(k) - \hat{\mathbf{y}}(k)||_F^2$$

### In practice

$$\mathbf{Y} = \left[ \begin{array}{cc} \hat{\mathcal{O}}_n & \mathbf{1}_N \end{array} \right] \left[ \begin{array}{c} B \\ D \end{array} \right]$$

# Time-domain identification (subspace method)

## Main properties

- Simple in parameterization
  No need for canonical forms for MIMO process models
  Subspace first, parameterization later
  Compact models in minimal realization
- Numerical property
  No nonlinear optimization techniques required
- Statistical property
  Simple Kalman filter framework

# Time-domain identification (subspace method)

## Algorithm & implementation

Input $\{t_i, \mathbf{u}_i, \mathbf{y}_i)\}_{i=1}^N \in \mathbb{R}$, $\texttt{tol} \in \mathbb{R}$

1. Compute Markov parameters $H_i\mathbb{R}$
2. Construct Hankel matrix $\mathcal{H}_n$
3. Estimate $r = \mathbf{rank}\ (\mathcal{H}_n)$
4. Apply `svd` and obtain projector $U_1$ and singular scalings $S_1$
5. Construct the estimated observability matrix $\hat{\mathcal{O}}_n$
6. Estimate $\hat{A}$ and $\hat{C}$
7. Estimate $\hat{B}$ and $\hat{D}$

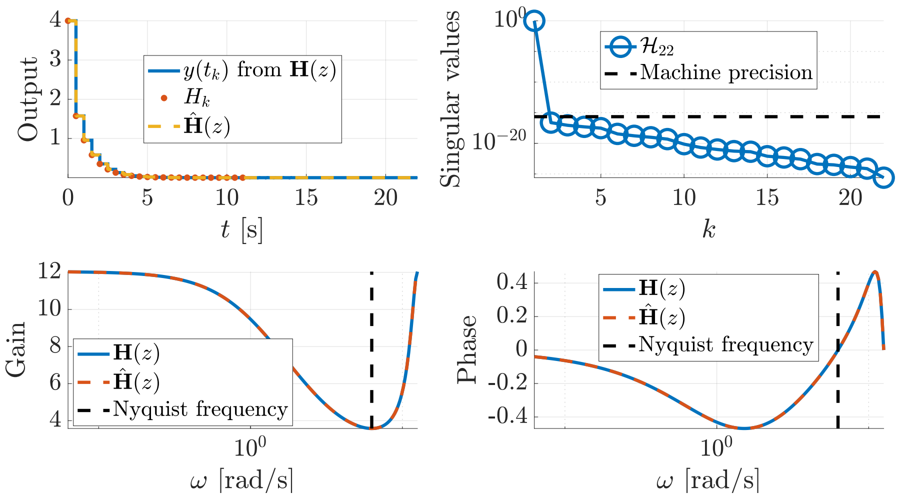
Output State-space model $\mathbf{H}$ and $\mathcal{S}$

**Implementation**

Subspace Framework, by MATLAB (with System Identification Toolbox)
`n4sid`

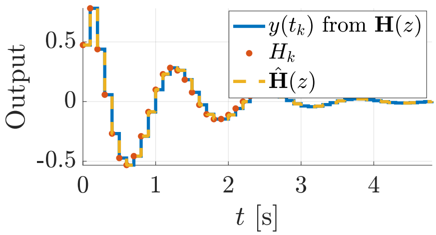
# Time-domain identification (subspace method)

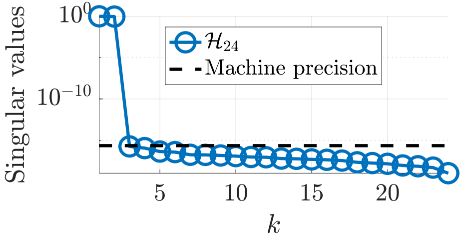
## Identification examples

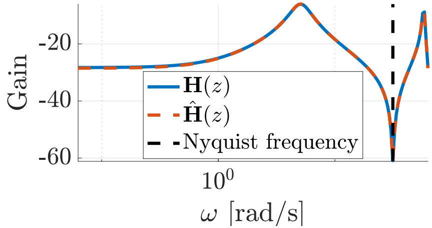


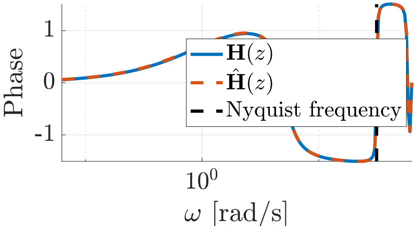
# Time-domain identification (subspace method)

## Identification examples



# Content

### SISO interpolation problem

Given the right and left data ($\lambda_j$ and $\mu_i$ are distinct):

$$\begin{array}{ll} \{\lambda_j, \mathbf{w}_j\} & j = 1, \ldots, k \\ \{\mu_i, \mathbf{v}_i^\top\} & i = 1, \ldots, q \end{array}$$

we seek $\mathcal{S} : (E, A, B, C)$, whose transfer function is $\mathbf{H}(s) = C(sE - A)^{-1}B$ s.t.

$$\begin{array}{rcll} \mathbf{H}(\lambda_j) & = & \mathbf{w}_j & j = 1, \ldots, k \\ \mathbf{H}(\mu_i) & = & \mathbf{v}_i^\top & i = 1, \ldots, q \end{array}$$

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A.J. Mayo and A.C. Antoulas, *"A framework for the solution of the generalized realization problem"*, Linear Algebra and its Applications, vol. 425(2-3), 2007.

I.V. Gosea, C. P-V. and A.C. Antoulas, *"Data-driven modeling and control of large-scale dynamical systems in the Loewner framework"*, Handbook in Numerical Analysis, vol. 23, January 2022.

# Frequency-domain identification (Loewner Framework)

## Loewner pencil (SISO)

The right data can be expressed as:

$$\begin{array}{rcl}\mathbf{\Lambda} & = & \mathbf{diag}\left[\lambda_1,\dots,\lambda_k\right] \in \mathbb{C}^{k\times k},\\ \mathbf{R} & = & \begin{bmatrix}1 & 1 & \dots & 1\end{bmatrix} \in \mathbb{C}^{1\times k}\\ \mathbf{W} & = & \begin{bmatrix}\mathbf{w}_1 & \mathbf{w}_2 & \dots & \mathbf{w}_k\end{bmatrix} \in \mathbb{C}^{1\times k}\end{array}$$

and the left data can be expressed as:

$$\begin{array}{rcl}\mathbf{M} & = & \mathbf{diag}\left[\mu_1,\dots,\mu_q\right] \in \mathbb{C}^{q\times q}\\ \mathbf{L}^\top & = & \begin{bmatrix}1 & 1 & \dots & 1\end{bmatrix} \in \mathbb{C}^{1\times q}\\ \mathbf{V}^\top & = & \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_q\end{bmatrix} \in \mathbb{C}^{1\times q}\end{array}$$

The Loewner matrix in this case is

$$\mathbb{L} = \begin{bmatrix}\frac{\mathbf{v}_1^\top - \mathbf{w}_1}{\mu_1 - \lambda_1} & \dots & \frac{\mathbf{v}_1^\top - \mathbf{w}_k}{\mu_1 - \lambda_k}\\ \vdots & \ddots & \vdots\\ \frac{\mathbf{v}_q^\top - \mathbf{w}_1}{\mu_q - \lambda_1} & \dots & \frac{\mathbf{v}_q^\top - \mathbf{w}_k}{\mu_q - \lambda_k}\end{bmatrix} \in \mathbb{C}^{q\times k}$$

With this notation $\mathbb{L}$ satisfy the Sylvester equation: $\mathbf{M}\mathbb{L} - \mathbb{L}\mathbf{\Lambda} = \mathbf{V}\mathbf{R} - \mathbf{L}\mathbf{W}$.

# Frequency-domain identification (Loewner Framework)

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$$\mathbf{M}\mathbb{L} - \mathbb{L}\mathbf{\Lambda} = \mathbf{V}\mathbf{R} - \mathbf{L}\mathbf{W}$$

The shifted Loewner matrix is:

$$\mathbb{S} = \begin{bmatrix} \frac{\mu_1 \mathbf{v}_1^\top - \mathbf{w}_1 \lambda_1}{\mu_1 - \lambda_1} & \cdots & \frac{\mu_1 \mathbf{v}_1^\top - \mathbf{w}_k \lambda_k}{\mu_1 - \lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mu_q \mathbf{v}_q^\top - \mathbf{w}_1 \lambda_1}{\mu_q - \lambda_1} & \cdots & \frac{\mu_q \mathbf{v}_q^\top - \mathbf{w}_k \lambda_k}{\mu_q - \lambda_k} \end{bmatrix} \in \mathbb{C}^{q \times k}$$

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# Frequency-domain identification (Loewner Framework)

## Loewner pencil (SISO)

If data are sampled from $\mathbf{G}(s) = C(sE - A)^{-1}B$, let us define :

$$\mathcal{O}_q = \begin{bmatrix} C(\mu_1 E - A)^{-1} \\ \vdots \\ C(\mu_q E - A)^{-1} \end{bmatrix} \in \mathbb{C}^{q \times n}, \quad \mathcal{R}_k = \left[ (\lambda_1 E - A)^{-1}B, \cdots, (\lambda_k E - A)^{-1}B, \right] \in \mathbb{C}^{n \times k}$$

be the generalized tangential observability and controllability matrices. Then,

$$\begin{aligned} [\mathbb{L}]_{ij} &= \frac{\mathbf{v}_i^\top - \mathbf{w}_j}{\mu_i - \lambda_j} \\ &= -C(\mu_j E - A)^{-1} E (\lambda_i E - A)^{-1} B \\ &= -[\mathcal{O}_q]_i E [\mathcal{R}_k]_j \end{aligned}$$

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# Frequency-domain identification (Loewner Framework)

## Realization and minimality

Assume that $k = q$, then $\mathbf{H}(s) = C(sE - A)^{-1}B$ with

$$E = -\mathbb{L}, \quad A = -\mathbb{S}, \quad , \; B = \mathbf{V}, \quad C = \mathbf{W},$$

is **a descriptor realization** interpolating the data.

Suppose that we have more data than necessary. The problem has a solution if

$$\mathbf{rank}\;[\xi\mathbb{L} - \mathbb{S}] = \mathbf{rank}\;[\mathbb{L},\; \mathbb{S}] = \mathbf{rank}\;\begin{bmatrix}\mathbb{L}\\ \mathbb{S}\end{bmatrix} = r, \quad \xi \in \{\lambda_i\} \cup \{\mu_j\}$$

$$[\mathbb{L},\; \mathbb{S}] = Y\Sigma_l\tilde{X}^\top, \quad \begin{bmatrix}\mathbb{L}\\ \mathbb{S}\end{bmatrix} = \tilde{Y}\Sigma_r X^H, \; Y, X \in \mathbb{C}^{N\times n}.$$

A realization $(E, A, B, C)$ of an (approximate) interpolant is given by:

$$E = -Y^H\mathbb{L}X, \quad A = -Y^H\mathbb{S}X, B = -Y^H\mathbf{V}, C = \mathbf{W}X$$

# Frequency-domain identification (Loewner Framework)

## Realization and minimality

Assume that $k = q$, then $\mathbf{H}(s) = C(sE - A)^{-1}B$ with

$$E = -\mathbb{L}, \quad A = -\mathbb{S}, \quad , \; B = \mathbf{V}, \quad C = \mathbf{W},$$

is **a descriptor realization** interpolating the data.

Suppose that we have more data than necessary. The problem has a solution if

$$\mathbf{rank}\ [\xi\mathbb{L} - \mathbb{S}] = \mathbf{rank}\ [\mathbb{L},\ \mathbb{S}] = \mathbf{rank}\ \begin{bmatrix} \mathbb{L} \\ \mathbb{S} \end{bmatrix} = r, \quad \xi \in \{\lambda_i\} \cup \{\mu_j\}$$

$$[\mathbb{L},\ \mathbb{S}] = Y\Sigma_l \tilde{X}^\top, \quad \begin{bmatrix} \mathbb{L} \\ \mathbb{S} \end{bmatrix} = \tilde{Y}\Sigma_r X^H, \; Y, X \in \mathbb{C}^{N\times n}.$$

A realization $(E, A, B, C)$ of an (approximate) interpolant is given by:

$$E = -Y^H\mathbb{L}X, \quad A = -Y^H\mathbb{S}X, B = -Y^H\mathbf{V}, C = \mathbf{W}X$$

# Frequency-domain identification (Loewner Framework)

## Main properties

Given $\{\lambda_j, \mathbf{r}_j, \mathbf{w}_j\}$ and $\{\mu_i, \mathbf{l}_i, \mathbf{v}_i\}$, seek $\mathbf{H}$ s.t.

$$\mathbf{H}(\lambda_j)\mathbf{r}_j = \mathbf{w}_j \text{ and } \mathbf{l}_i\mathbf{H}(\mu_i) = \mathbf{v}_i$$

$j = 1, \ldots, k$; $i = 1, \ldots, q$.

Rational interpolation

$$\mathbf{H}(s) = \mathbf{W}(-s\mathbb{L} + \mathbb{S})^{-1}\mathbf{V}$$

A.C. Antoulas, S. Lefteriu and A.C. Ionita, *"Chapter 8: A Tutorial Introduction to the Loewner Framework for Model Reduction"*, Model Reduction and Approximation: Theory and Algorithms, 2016.

# Frequency-domain identification (Loewner Framework)

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Rational interpolation
$\mathbf{H}(s) = \mathbf{W}(-s\mathbb{L} + \mathbb{S})^{-1}\mathbf{V}$

- underlying rational $(r)$ order

$$\begin{array}{rcl} r & = & \textbf{rank}\ (\xi\mathbb{L} - \mathbb{S}) \\ & = & \textbf{rank}\ ([\mathbb{L}, \mathbb{S}]) \\ & = & \textbf{rank}\ ([\mathbb{L}^H, \mathbb{S}^H]^H) \end{array}$$

- and McMillan $(\nu)$ order

$$\nu \quad = \quad \textbf{rank}\ (\mathbb{L})$$

- $\mathbb{L}$ and $\mathbb{S}$ are input-output independents.
- Minimal realization

A.C. Antoulas, S. Lefteriu and A.C. Ionita, *"Chapter 8: A Tutorial Introduction to the Loewner Framework for Model Reduction"*, Model Reduction and Approximation: Theory and Algorithms, 2016.

# Frequency-domain identification (Loewner Framework)

## Algorithm & implementation

Input $\{t_i, \mathbf{u}_i, \mathbf{y}_i)\}_{i=1}^N \in \mathbb{R}$, $\texttt{tol} \in \mathbb{R}$

1. Compute FRF $\{\imath\omega_i, \mathbf{G}(\imath\omega_i)\}_{i=1}^N \in \mathbb{C}$
2. Select the column and row variables: $\{\mathbf{\Lambda}, \mathbf{W}, \mathbf{R}\}$ and $\{\mathbf{M}, \mathbf{V}, \mathbf{L}\}$
3. Construct $\mathbb{L}$, $\mathbb{S}$ , $\mathbf{V}$ and $\mathbf{W}$ (full realization)
4. Estimate $\nu = \textbf{rank}\ (\mathbb{L})$, $r = \textbf{rank}\ ([\mathbb{L}\ \mathbb{S}])$
5. Apply `svd` and obtain projectors $X$ and $Y$
6. Project to get minimal or reduced-order realization

Output State-space model $\mathbf{H}$ and $\mathcal{S}$

**Implementation**

Loewner Framework (tangential version), by pedagogical team

`insapack.non_param_freq` $\rightarrow$ `insapack.data2loewner` $\rightarrow$ `insapack.loewner_tng`

# Frequency-domain identification (Loewner Framework)

## Identification examples (simple case)

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

$$\mathbf{G}(s) = \frac{1}{s^2+1}$$

Evaluated at

$$\lambda_1 = 1, \lambda_2 = 2, \mu_1 = -1, \mu_2 = -2$$

Leads to

$$\mathbf{w}_1 = \frac{1}{2}, \mathbf{w}_2 = \frac{1}{5}, \mathbf{v}_1 = \frac{1}{2}, \mathbf{v}_2 = \frac{1}{5}$$

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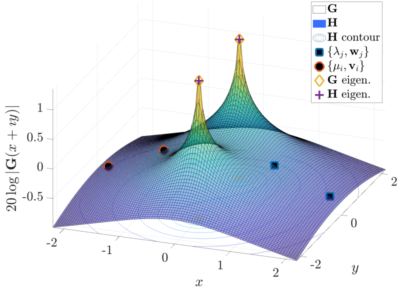


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$$\mathbf{W} = \begin{bmatrix} \frac{1}{2} & \frac{1}{5} \end{bmatrix}, \mathbf{V} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{5} \end{bmatrix}$$

$$\mathbb{L} = \begin{bmatrix} 0 & -\frac{1}{10} \\ \frac{1}{10} & 0 \end{bmatrix}, \mathbb{S} = \begin{bmatrix} \frac{1}{2} & \frac{3}{10} \\ \frac{3}{10} & \frac{1}{5} \end{bmatrix}$$

# Frequency-domain identification (Loewner Framework)

## Identification examples (simple case)

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

Rank reveals the underlying rational ($r$) and McMillan ($\nu$) orders

$$\textbf{rank } (\xi\mathbb{L} - \mathbb{S}) = r$$

$$\textbf{rank } (\mathbb{L}) = \nu$$

$r = 2$ and $\nu = 2$, $(\mathbb{S}, \mathbb{L})$ pencil regular

$$\mathbf{H}(s) = \mathbf{W}(-s\mathbb{L} + \mathbb{S})^{-1}\mathbf{V} = \mathbf{G}(s)$$

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# Frequency-domain identification (Loewner Framework)

## Identification examples (lot of data case)

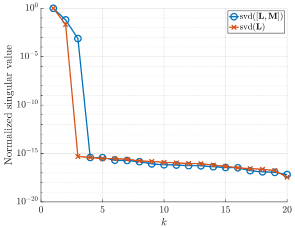
Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

$$\mathbf{G}(s) = \frac{s^2 + s + 2}{s + 1}$$

Evaluated at

$$\lambda_{1\ldots 20} = [1, 2, \ldots, 20]$$
$$\mu_{1\ldots 20} = [1.5, 2.5, \ldots, 20.5]$$



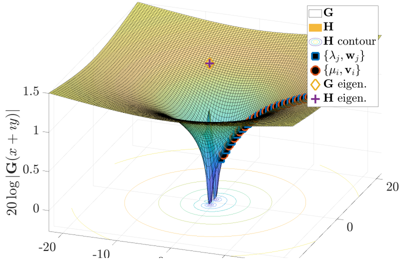
# Frequency-domain identification (Loewner Framework)

## Identification examples (lot of data case)

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

Realization $n = 20$:
$\mathbf{H}(s) = \mathbf{W}(-s\mathbb{L} + \mathbb{S})^{-1}\mathbf{V}$




$$\mathbf{G}(s) = \frac{s^2 + s + 2}{s + 1}$$

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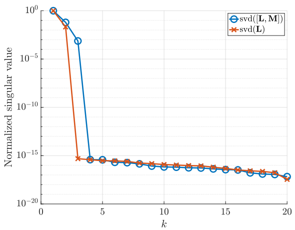

# Frequency-domain identification (Loewner Framework)

## Identification examples (lot of data case)

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$$\textbf{rank } (\xi\mathbb{L} - \mathbb{S}) = r = 3$$

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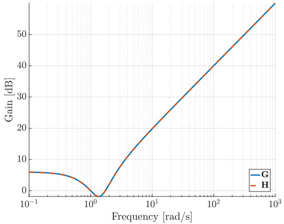

# Frequency-domain identification (Loewner Framework)

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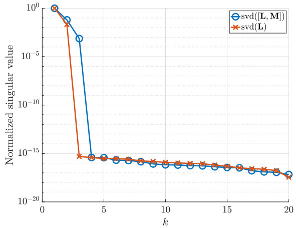
$$\begin{aligned}\mathbf{H}(s) &= \mathbf{W}X(-sY^\top\mathbb{L}X + Y^\top\mathbb{S}X)^{-1}Y^\top\mathbf{V}\\ &= \frac{s^2+s+2}{s+1}\end{aligned}$$

$$\mathbf{G}(s) = \frac{s^2+s+2}{s+1}$$

Evaluated at

$$\lambda_{1\ldots 20} = [1, 2, \ldots, 20]$$
$$\mu_{1\ldots 20} = [1.5, 2.5, \ldots, 20.5]$$



# Content

# Conclusion and closing example

**Example (if time)**

# Linear dynamical system identification

... basic elements and Labs guidelines

Charles Poussot-Vassal

February 20, 2026