# Linear dynamical system identification

... basic elements and Labs guidelines

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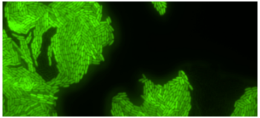
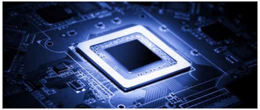
February 20, 2026

INSTITUT NATIONAL DES SCIENCES APPLIQUÉES TOULOUSE

### Dynamical models are centrals tools in (control) engineering

- **for verification and validation**
  ($\mu$, $\mathcal{H}_\infty$-norm, pseudo-spectra, Monte Carlo)
- **for detection**
  (fault isolation, param. & variables estim.)
- **for uncertainty propagation & detection**
  (Multi Disc. Optim., robust optim.)
- **for feedback control synthesis**
  ($\mathcal{H}_\infty/\mathcal{H}_2$-norm, MPC, adaptive)
- **for system optimization**
  (many-query simulation-based)





## Motivations (antenna and telecommunication problem)

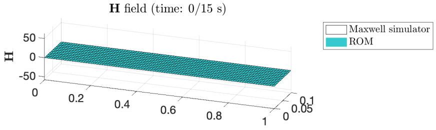
### Antenna, cellular...

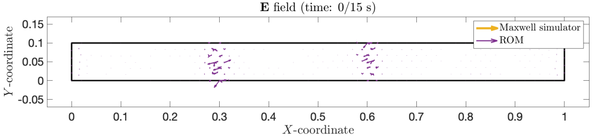
**Antenna models**

- **to optimize parameters**
- **for polar computation**

**Blend physics from**

- **Maxwell equations**
- **Kirchoff equations**




- Replace costly simulations by accurate simple model
- Preserve structure and properties (port-Hamiltonian)
- Allows for geometry optimization

M. Gouzien, C. P-V., G. Haine and D. Matignon, *"A Port-Hamiltonian reduced order modelling of the 2D Maxwell equations"*, journal for Computation and Mathematics in Electrical and Electronic Engineering, 2025.

# Introduction

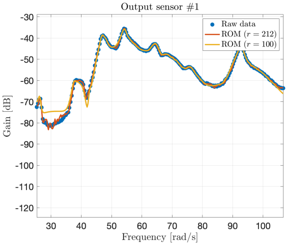
## Motivations (Dassault-Aviation ground vibration experiments)

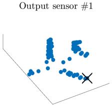
**Aircraft models used**

- for control design
- for life-cycle monitoring

**Physics involved**

- structure
- sensing



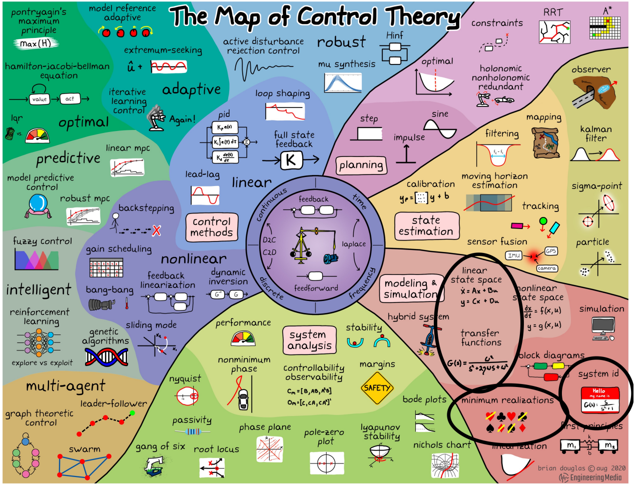

### Raw data vs. Reduced model

- Digital twin of the real aircraft with $\approx 100$ outputs
- Process employed in ground & flight tests

C. P-V. and P. Vuillemin, *"MDSPACK"*, MOR Digital Systems.

# Introduction

## The map of control theory (by Brian Douglas - `https://engineeringmedia.com/`)



# Introduction

## Course plan, overview and questions addressed

### L1: Overview, signals construction, pre-treatment and non-parametric analysis

- Realization, transfer functions, ...
- Construct an experimental plan and signals.
- How to analyze signals, and derive some properties?

### L2: Data-driven model construction in the time- and frequency-domain

- Construct a linear model from time- or frequency-domain data.
- How much is it valid? How to validate, discuss, amend it?

### L3: L2 cont'd & Labs guidelines

- Illustration in practice
- Experimental setup & numerical tools presentation.
- Methodology for the lab.

### TP: autonomous labs

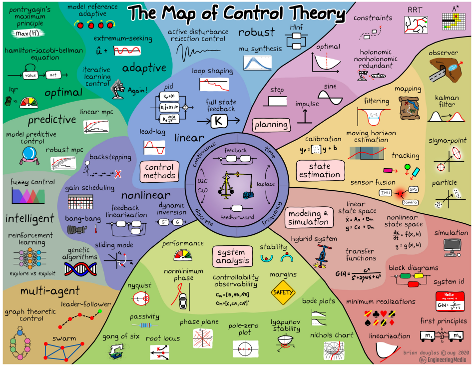
- 5 sessions, 2h45 each
- Group of 3 students

### Motto

Explore dyn. systems & control engineering

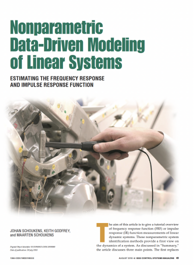
- **Identification**
  **Modeling**
  **Analysis**
  Control
  Performance optimisation
- Team group
  Team work & presentation

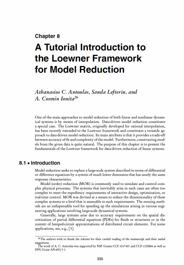
The map of control theory, by Brian Douglas
`https://engineeringmedia.com/`

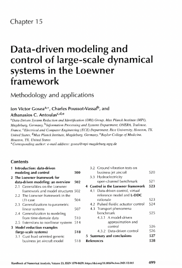


# Introduction

## References and material




https://github.com/cpoussot/insapack

- Signals
- Non-parametric analysis
- Identification TD
- Identification FD
- Examples
- Codes

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J. Schoukens, K. Godfrey, M. Schoukens, *"Nonparametric data-driven modeling of linear systems"*, IEEE Control System Magazine, 2018.

A.C. Antoulas, A.C. Ionita, S. Lefteriu, *"A Tutorial Introduction to the Loewner Framework for Model Reduction"*, Model Reduction and Approximation, Ch. 8.

I.V. Gosea, C. P-V., A.C. Antoulas,, *"Data-driven modeling and control of large-scale dynamical systems in the Loewner framework"*, Handbook on Numerical Analysis, Ch. 11.

**L1: Overview, signals construction, pre-treatment and non-parametric analysis**

- Realization, transfer functions, ...
- Construct an experimental plan and signals.
- How to analyze signals, and derive some properties?

### Notions treated

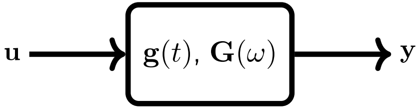
- Linear models: Transfer Function (TF), realization
- Observability, controllability, Hankel, minimality
- Fourier Transform (FT), Frequency Response Function (FRF), Impulse Response (IR)
- Measurement setup
- Binary sequence, periodic signals
- Averaging, mean, variance

# Content

### Model

(Internal representation - time-domain) $\mathbf{H} \sim \mathbf{u} \rightarrow \mathbf{x} \rightarrow \mathbf{y}$

(External representation - frquency-domain) $\mathbf{H} \sim \mathbf{u} \rightarrow \mathbf{y}$




(time-domain) $\{t_k, \mathbf{u}(t_k), \mathbf{y}(t_k)\}_{k=1}^N$

(frequency-domain) $\{\omega_k, \mathbf{U}(\omega_k), \mathbf{Y}(\omega_k)\}_{k=1}^N$

### Data

**Notations**

$\mathbf{u}(t_k)$ and $\mathbf{y}(t_k)$

$\mathbf{U}(\omega_k)$ and $\mathbf{Y}(\omega_k)$

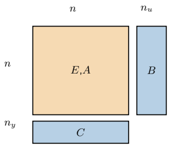
where $k = 1, \ldots, N$

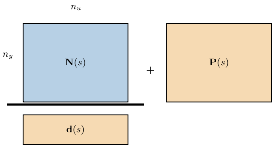
$$\left\{\begin{array}{lcl} t_k & = & kT_s \\ \omega_k & = & 2\pi f_k \\ f_k & = & kf_0 = k\dfrac{f_s}{N} \\ f_s & = & \dfrac{1}{T_s} \end{array}\right.$$

- $f_0$: frequency resolution
- $N$: number of samples

# Reminder on linear dynamical models

## Realization and transfer function (generic form)




$$\mathbf{H}:\begin{cases} E\dot{\mathbf{x}}(t) &= A\mathbf{x}(t)+B\mathbf{u}(t) \\ \mathbf{y}(t) &= C\mathbf{x}(t)\end{cases}$$

$$\begin{aligned}\mathbf{H}(s) &= C\left(sE-A\right)^{-1}B \\ &= \mathbf{N}(s)/\mathbf{d}(s)+\mathbf{P}(s)\end{aligned}$$

### Realization

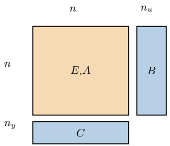
- $E, A, B, C$ are matrices
- Internal knowledge $\mathbf{u} \mapsto \mathbf{x} \mapsto \mathbf{y}$
- Infinite number of realizations
- $\mathbf{u}(t) \in \mathbb{R}^{n_u}$,
  $\mathbf{y}(t) \in \mathbb{R}^{n_y}$,
  $\mathbf{x}(t) \in \mathbb{R}^{n}$

### Transfer function

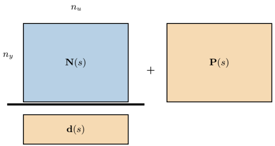
- $\mathbf{H}$ is a complex function
- External knowledge $\mathbf{u} \mapsto \mathbf{y}$
- Unique function
- $\mathbf{U}(s) \in \mathbb{C}^{n_u}$,
  $\mathbf{Y}(s) \in \mathbb{C}^{n_y}$

# Reminder on linear dynamical models

## Realization and transfer function (generic form)




$$\mathbf{H} : \begin{cases} E\dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + B\mathbf{u}(t) \\ \mathbf{y}(t) &= C\mathbf{x}(t) \end{cases}$$




$$\begin{aligned} \mathbf{H}(s) &= C\left(sE - A\right)^{-1}B \\ &= \mathbf{N}(s)/\mathbf{d}(s) + \mathbf{P}(s) \end{aligned}$$

### Realization

- $E, A, B, C$ are matrices
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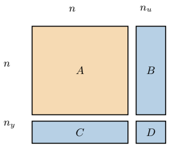
  $\mathbf{x}(t) \in \mathbb{R}^{n}$

### Transfer function

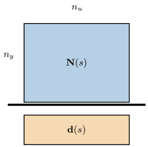
- $\mathbf{H}$ is a complex function
- External knowledge $\mathbf{u} \mapsto \mathbf{y}$
- Unique function
- $\mathbf{U}(s) \in \mathbb{C}^{n_u}$,
  $\mathbf{Y}(s) \in \mathbb{C}^{n_y}$

# Reminder on linear dynamical models

## Realization and transfer function (ODE & bounded)




$$\mathbf{H} : \begin{cases} \dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + B\mathbf{u}(t) \\ \mathbf{y}(t) &= C\mathbf{x}(t) + D\mathbf{u}(t) \end{cases}$$




$$\begin{aligned} \mathbf{H}(s) &= C\big(sE - A\big)^{-1}B + D \\ &= \mathbf{N}(s)/\mathbf{d}(s) \end{aligned}$$

### Realization

- $A, B, C,$ **D** are **real** matrices
- Internal knowledge $\mathbf{u} \mapsto \mathbf{x} \mapsto \mathbf{y}$
- Infinite number of realizations
- $\mathbf{u}(t) \in \mathbb{R}^{n_u}$,
  $\mathbf{y}(t) \in \mathbb{R}^{n_y}$,
  $\mathbf{x}(t) \in \mathbb{R}^{n}$

### Transfer function

- $\mathbf{H}$ is a **rational** complex function
- External knowledge $\mathbf{u} \mapsto \mathbf{y}$
- Unique function
- $\mathbf{U}(s) \in \mathbb{C}^{n_u}$,
  $\mathbf{Y}(s) \in \mathbb{C}^{n_y}$

# Reminder on linear dynamical models

## Finite dimensional... some differences

**Structures**

- **L-ODE**
- L-ODE / DAE-1
- L-DAE

Transfer function

$$\mathbf{H}(s) = \frac{2}{s+1}$$

ODE realization $\mathbf{H}$

$$\begin{array}{rcl} \dot{x} & = & -x + 2u \\ y & = & x \end{array}$$

Singularities $\lambda$ and zeros $z$

$$\begin{array}{rcl} \lambda_{\mathbf{H}} & = & \texttt{eig(A,E)} \\ & = & \Lambda(-1, 1) \\ & = & \{-1\} \\ z_{\mathbf{H}} & = & \texttt{eig([A B; C D],blkdiag(E,zeros(ny,nu)))} \\ & = & \Lambda\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \\ & = & \{\infty, \infty\} \end{array}$$

# Reminder on linear dynamical models

## Finite dimensional... some differences

**Structures**

- L-ODE
- **L-ODE / DAE-1**
- L-DAE

Transfer function

$$\mathbf{H}(s) = \frac{2}{s+1} + 2 = \frac{2s+4}{s+1}$$

ODE realization $\mathbf{H}_1$

$$\begin{array}{rcl} \dot{x} & = & -x + 2u \\ y & = & x + 2u \end{array}$$

Singularities of matrix pencil $(A, E)$

$$\lambda_{\mathbf{H}_1} = \Lambda(-1, 1) = \{-1\}$$

# Reminder on linear dynamical models

## Finite dimensional... some differences

**Structures**

- L-ODE
- **L-ODE / DAE-1**
- L-DAE

Transfer function

$$\mathbf{H}(s) = \frac{2}{s+1} + 2 = \frac{2s+4}{s+1}$$

DAE index-1 realization $\mathbf{H}_2$

$$\begin{array}{rcl} \dot{x}_1 & = & -x_1 + 2u \\ 0 & = & -x_2 + 2u = x_2 - 2u \\ y & = & x_1 + x_2 \end{array}$$

Singularities of matrix pencil $(A, E)$[a]

$$\lambda_{\mathbf{H}_2} = \Lambda\left( \left[\begin{array}{c|c} -1 & \\ \hline & 1 \end{array}\right], \left[\begin{array}{c|c} 1 & \\ \hline & \end{array}\right] \right) = \{-1, \infty\}$$

[a] $B^\top = \begin{bmatrix} 2 & -2 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 1 \end{bmatrix}$

# Reminder on linear dynamical models

## Finite dimensional... some differences

**Structures**

- L-ODE
- **L-ODE / DAE-1**
- L-DAE

Transfer function

$$\mathbf{H}(s) = \frac{2}{s+1} + 2 = \frac{2s+4}{s+1}$$

DAE index-1 realization $\mathbf{H}_2$ (canonical form)

$$\left( \left[ \begin{array}{c|c} A_1 = -1 & \\ \hline & I_{n_2} = 1 \end{array} \right], \left[ \begin{array}{c|c} I_{n_1} = 1 & \\ \hline & N = 0 \end{array} \right] \right)$$

Index is the $k$-nilpotent degree of $N$

- Finite dynamic modes
  $n_1 = 1$
- Infinite dynamic (impulsive) modes
  $\mathbf{rank}\ (E) - n_1 = 0$
- Non dynamic modes
  $n - \mathbf{rank}\ (E) = 1$

# Reminder on linear dynamical models

## Finite dimensional... some differences

**Structures**

- L-ODE
- L-ODE / DAE-1
- **L-DAE**

Transfer function

$$\mathbf{H}(s) = \frac{2}{s+1} + s = \frac{s^2+s+2}{s+1}$$

DAE index-2 realization $\mathbf{H}$

$$\begin{array}{rcl} \dot{x}_2 & = & x_1 \\ \dot{x}_3 & = & x_2 \\ x_2 & = & -x_3 + u = x_3 - u \\ y & = & x_1 + x_2 + 2x_3 \end{array}$$

Singularities of matrix pencil $(A, E)$

$$\lambda_{\mathbf{H}} = \{-1, \infty, \infty\}$$

Finite / impulsive & (non) dynamic modes $n_1 = 1$, $\textbf{rank}\,(E) - n_1 = 1$, $n - \textbf{rank}\,(E) = 1$

# Reminder on linear dynamical models

## Finite dimensional... some differences

**Structures**

- L-ODE
- L-ODE / DAE-1
- **L-DAE**

Transfer function

$$\mathbf{H}(s) = \frac{2}{s+1} + s = \frac{s^2+s+2}{s+1}$$

DAE index-2 realization $\mathbf{H}$

$$\begin{array}{rcl} \dot{x}_2 &=& x_1 \\ \dot{x}_3 &=& x_2 \\ x_2 &=& -x_3 + u = x_3 - u \\ y &=& x_1 + x_2 + 2x_3 \end{array}$$

Singularities of matrix pencil $(A, E)$

$$\lambda_{\mathbf{H}} = \{-1, \infty, \infty\}$$

Finite / impulsive & (non) dynamic modes $n_1 = 1$,
$\mathbf{rank}\ (E) - n_1 = 1$, $n - \mathbf{rank}\ (E) = 1$

# Reminder on linear dynamical models

## Observability, controllability, minimality...

**Structures**

- **L-ODE**
- **L-ODE / DAE-1**
- L-DAE

- Reachability matrix

$$\mathcal{R}_n(A,B) = \left[\begin{array}{cccccc} B & AB & A^2B & \cdots & A^nB \end{array}\right]$$

- Observability matrix

$$\mathcal{O}_n(A,C) = \left[\begin{array}{cccccc} C^\top & A^\top C^\top & (A^\top)^2C^\top & \cdots & (A^\top)^nC^\top \end{array}\right]^\top$$

- Minimality: both controllable and observable.
- Connection with Markov parameters

$$H_0 = D \ , \ H_k = CA^{k-1}B \ (k \geq 1)$$

$$\mathcal{H}_n = \mathcal{O}_n\mathcal{R}_n = \left[\begin{array}{cccc} H_1 & H_2 & \cdots & H_n \\ H_2 & H_3 & \cdots & H_{n+1} \\ \vdots & & \ddots & \vdots \\ H_n & H_{n+1} & \cdots & H_{2n-1} \end{array}\right]$$

# Reminder on linear dynamical models

## Example #1 (with $D$-term)

### Continuous-time

$$\mathbf{H}(s) = \frac{2s+4}{s+1}$$

$$\mathcal{S} : \left\{ \begin{array}{rcl} \dot{x}(t) & = & -x(t) + 2u(t) \\ y(t) & = & x(t) + 2u(t) \end{array} \right.$$

### Sampled-time

$$\mathbf{H}(z) = \frac{2z - 0.4261}{z - 0.6065}$$

$$\mathcal{S} : \left\{ \begin{array}{rcl} x(k+1) & = & 0.6065x(k) + u(k) \\ y(k) & = & 0.7869x(k) + 2u(k) \end{array} \right.$$

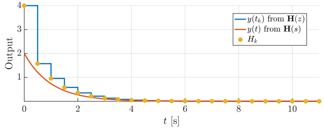
# Reminder on linear dynamical models

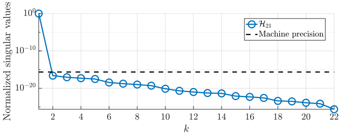
## Example #1 (with $D$-term)

### Continuous-time

$$\mathbf{H}(s) = \frac{2s+4}{s+1}$$

$$\mathcal{S}: \begin{cases} \dot{x}(t) &= -x(t) + 2u(t) \\ y(t) &= x(t) + 2u(t) \end{cases}$$




### Sampled-time

$$\mathbf{H}(z) = \frac{2z - 0.4261}{z - 0.6065}$$

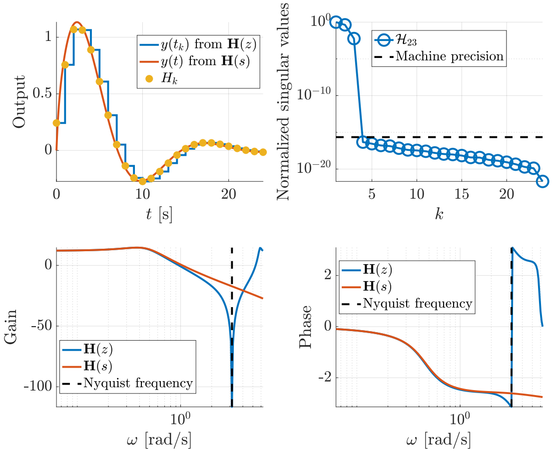
$$\mathcal{S}: \begin{cases} x(k+1) &= 0.6065x(k) + u(k) \\ y(k) &= 0.7869x(k) + 2u(k) \end{cases}$$

- $T_s = 0.5$, 'zoh'
- $n = 1$

- $\textbf{rank}\ (\mathcal{O}_{n+20}) = 1$
- $\textbf{rank}\ (\mathcal{R}_{n+20}) = 1$
- $\textbf{rank}\ (\mathcal{H}_{n+20}) = 1$

# Reminder on linear dynamical models

## Example #2 (higher order)



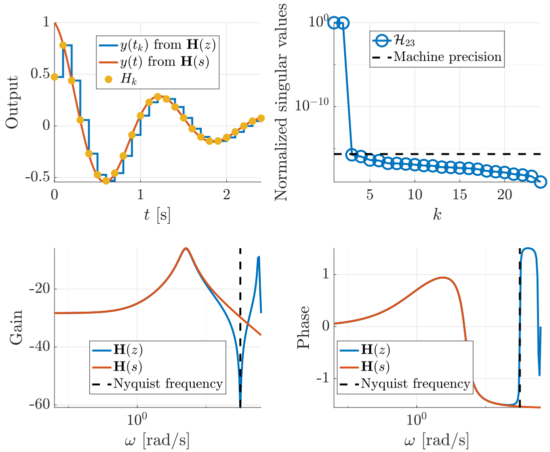

$$\mathbf{H}(s) = \frac{2s+4}{s^3+5s^2+2s+1}$$

- $T_s = 1$, `'tustin'`
- $n = 3$

- $\textbf{rank}\ (\mathcal{O}_{n+20}) = 3$
- $\textbf{rank}\ (\mathcal{R}_{n+20}) = 3$
- $\textbf{rank}\ (\mathcal{H}_{n+20}) = 3$

# Reminder on linear dynamical models

## Example #2 (partially observable & minimality)




$$A = \begin{bmatrix} -1 & 5 & \\ -5 & -1 & \\ & & -3 \end{bmatrix}$$

$$B, C^\top = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- $T_s = 0.1$, `'foh'`
- $n = 3$

- **rank** $(\mathcal{O}_{n+20}) = 2$
- **rank** $(\mathcal{R}_{n+20}) = 3$
- **rank** $(\mathcal{H}_{n+20}) = 2$

# Content

# Practical estimation from data

## Impulse response estimation ($\mathbf{g}$)

### Time-domain response

The response of the dynamical system is:

$$\mathbf{y}(t) = \int_{-\infty}^{\infty} \mathbf{g}(\tau)\mathbf{u}(t-\tau)d\tau = \mathbf{g}(t) * \mathbf{u}(t).$$

### Estimate IR, $\mathbf{g}(t)$

- Solve the deconvolution problem, or
- Use cross-correlation functions

$$\begin{array}{rcl} R_{\mathbf{yu}}(\tau) & = & \mathbf{g}(\tau) * R_{\mathbf{uu}}(\tau) \\ & = & \mathbf{g}(\tau)\sigma_{\mathbf{u}}^2 \end{array}$$

with white random noise

**Markov parameters and IR**

$$\mathbf{y}(t_k) = \begin{cases} D & \text{if } k = 0 \\ CA^{k-1}B & \text{if } k > 0 \end{cases}$$

**Cross-correlation**

$$\begin{array}{rcl} R_{\mathbf{yu}}(\tau) & = & E\{\mathbf{y}(t+\tau)\mathbf{u}(t)\} \\ R_{\mathbf{uu}}(\tau) & = & E\{\mathbf{u}(t+\tau)\mathbf{u}(t)\} \end{array}$$

# Practical estimation from data

## Frequency response estimation (G)

### Frequency-domain response

The response of the dynamical system is:

$$\mathbf{Y}(k) = \mathbf{G}(k)\mathbf{U}(k)$$

### Estimate FRF, $\mathbf{G}(k)$

- Apply (fails when $\mathbf{U}$ close to zero)

$$\mathbf{G}(k) = \mathbf{Y}(k)/\mathbf{U}(k)$$

- Use cross-correlation spectrum functions

$$S_{\mathbf{yu}}(k) = \mathcal{F}\left(R_{\mathbf{yu}}\right) \text{ and } S_{\mathbf{uu}}(k) = \mathcal{F}\left(R_{\mathbf{uu}}\right)$$

$$\mathbf{G}(k) = \frac{S_{\mathbf{yu}}(k)}{S_{\mathbf{uu}}(k)}$$

**Fourier transform**

Let $\mathbf{x}(t)$, $t = 0, \cdots, N-1$

$$\begin{aligned} \mathbf{X}(k) &= \frac{1}{\sqrt{N}} \sum_{t=0}^{N-1} \mathbf{x}(t) e^{-\imath 2\pi t k/N} \\ \mathbf{x}(t) &= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \mathbf{X}(t) e^{\imath 2\pi t k/N} \end{aligned}$$

$\mathbf{X}(k)$ is the Fourier coefficient of $\mathbf{x}(t)$ and frequency $f_k = k f_s/N$ ($f_s = 1/T_s$).

- FFT more accurate with power of 2

# Content

# Measurement setup

## From generator to measurement

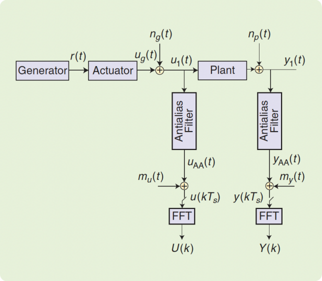


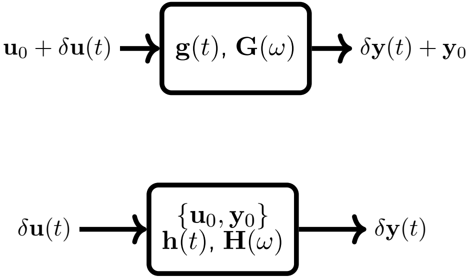

**FIGURE 2** Measurement setup and notations. The generator signal is disturbed by generator noise $n_g(t)$, and the output of the system is disturbed by the process noise $n_p(t)$. The measured input and output signals are first low-pass filtered by the antialias filters. The measurement noise on the input and output is, respectively, $m_u(t)$ and $m_y(t)$. These signals are sampled at rate $f_s = 1/T_s$. The discrete Fourier transform of the measurements $u(kT_s), y(kT_s), k = 1, \ldots, N$ is calculated using the fast Fourier transform (FFT) algorithm, and it is denoted as $U(k), Y(k)$. The frequency index $k$ indicates the frequency $kf_s/N$.

- Store:
  $\mathbf{r}(t)$: generated signal
  $\mathbf{u}_g(t)$: effectively sent signal
  $\mathbf{u}(t)$: plant input $\mathbf{y}(t)$: plant output
- FFT: more accurate when signals of length $N = 2^k$
- Discuss with experimental team about AAF
  IIR, FIR, delay, cut-off frequency?
- Discuss with experimental team about noises
  knowledge on input/output?


J. Schoukens, K. Godfrey, M. Schoukens, *"Nonparametric data-driven modeling of linear systems"*, IEEE Control System Magazine, 2018.

# Measurement setup

## Working point




### The practical setup

A system evolves

- around an equilibrium point
- along trajectories

### The theoretical model

A linear model $\mathbf{H}$ is valid

- around an equilibrium point

$$\{\mathbf{u}_0, \mathbf{y}_0\}$$

- along trajectories[a]

$$\{\mathbf{u}(t,p), \mathbf{y}(t,p)\}$$

[a] $t$: time, $p$: parameter

# Content

# Experiment design: choice of the exciting signal

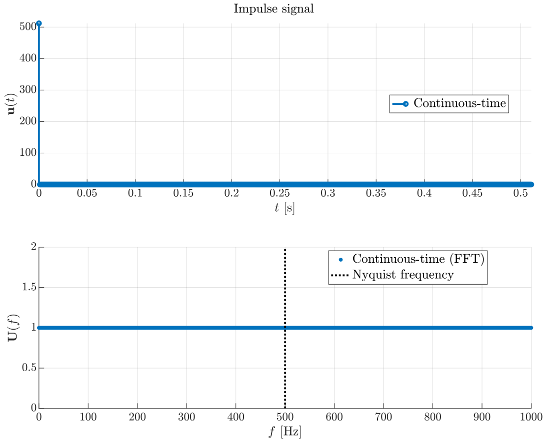
## Impulse signals

$$\mathbf{u}(t) = \begin{cases} N_s & \text{if } t = 0 \\ 0 & \text{otherwise} \end{cases}$$

where, $N_s = 2^9$

**Characteristics**

+ White noise
− Not easily applicable in practice



# Experiment design: choice of the exciting signal

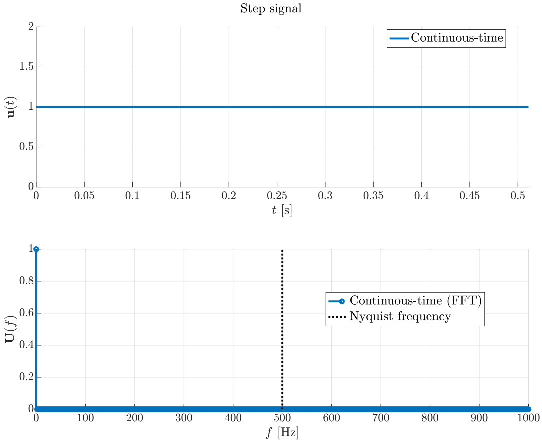
## Step signals

$$\mathbf{u}(t) = 1$$

where, $N_s = 2^9$

**Characteristics**

+ Easily applicable in practice
– Spectrum not rich



# Experiment design: choice of the exciting signal

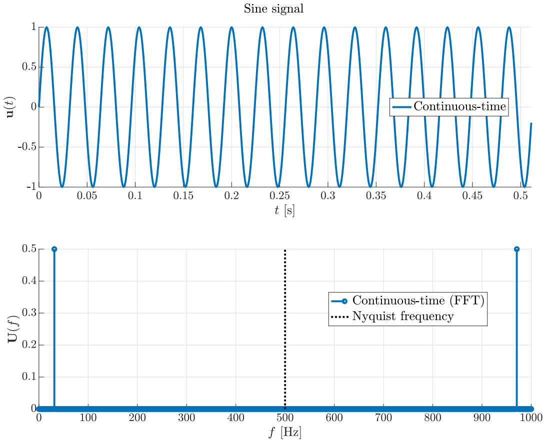
## Sine signals

$$\mathbf{u}(t) = \sin(2\pi f_1 t)$$

where, $N_s = 2^9$

### Characteristics

+ Easily applicable in practice
+ Provides local information
– Spectrum not rich



# Experiment design: choice of the exciting signal

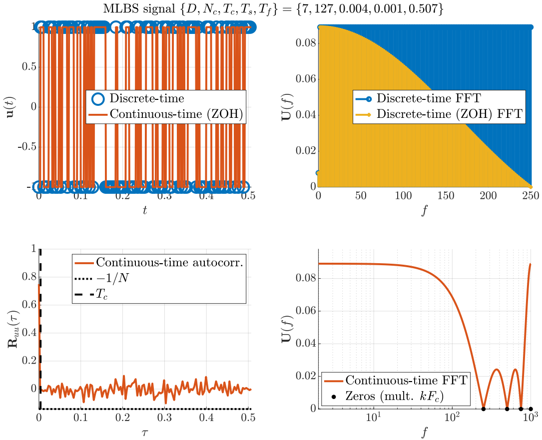
## Maximum Length Binary Sequence signals

$$\mathbf{u}(t) = \{-1, +1\}$$

$N_s = 2^9 \Rightarrow$
$f_c = 4f_{max}$ & $N_s = 508$

### Characteristics

- $+$ Easily applicable in practice
- $+$ Provides rich information
- $-$ Sensitive to nonlinear distortions (this can be attenuated by randomized realization or inversing $\{+1, -1\}$)
- ▶ `insapack.mlbs`



# Experiment design: choice of the exciting signal

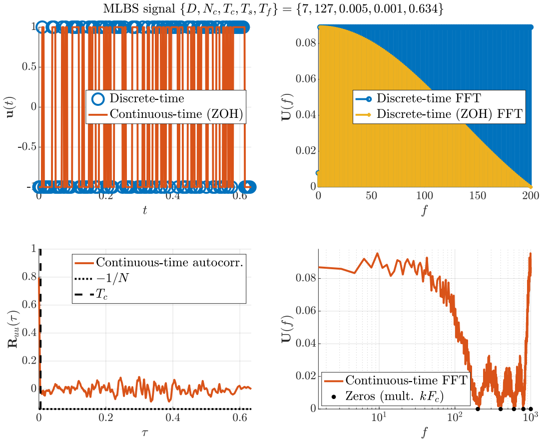
## Maximum Length Binary Sequence signals

$$\mathbf{u}(t) = \{-1, +1\}$$

$N_s = 2^9 \Rightarrow$
$f_c = 5 f_{max}$ & $N_s = 635$

**Guidelines**

- Use $f_c = \frac{1}{T_c} > 2.5 f_{max}$ to obtain a sufficiently flat amplitude in the frequency range
- Signal length $N_s$, such that resolution $f_0 = \frac{1}{N_s}$ meets requierements
- Do not modify by zero padding, instead use generalized FFT should be used



# Experiment design: choice of the exciting signal

## Maximum Length Binary Sequence signals (`insapack.mlbs`)

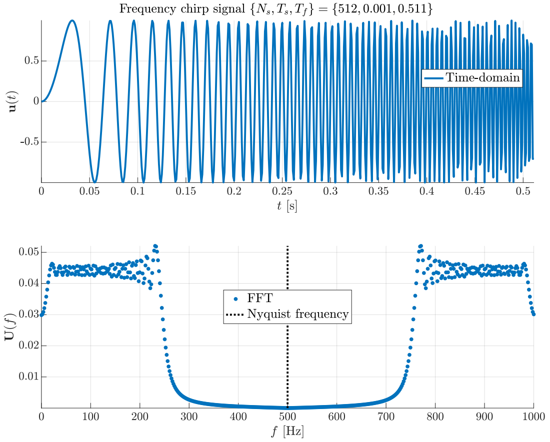
```
% Description
% Computes a MLBS signal over FBND frequency range.
%
% Syntax
%  [uc,tc,info] = insapack.mlbs(Ns,Ts,FBND,REV,SHOW)
%
% Input arguments
%  - Ns   : number of samples - estimated (integer)
%           Ns may be modified to ensure the number of MLBS cell generating
%           is a 2^D-1 sequence, where D = round(log2(N*Ts/Tc))
%  - Ts   : sampling period [s] (>0 real)
%  - FBND : frequency range [Hz] (2x1 real vector)
%  - REV  : reverse signal (boolean)
%           - false: fmin to max
%           - true: fmax to fmin
%  - SHOW : plot signal (boolean)
%
% Output arguments
%  - uc   : chirp signal in {-1,1}
%  - tc   : time samples
%  - info : additional information
%
% See also for the Polynomial feedback sequence and explanations on MLBS
% and white noise generation:
% https://www.digikey.fr/fr/articles/techzone/2018/mar/use-readily-available-components-generate-binary-sequence
```

# Experiment design: choice of the exciting signal

## Chirp signals

$$\mathbf{u}(t) = A \sin\left(2\pi\left(\frac{\beta t^{1+p}}{1+p} + f_0 t\right)\right)$$

$\beta = (f_1 - f_0)t^{-p}$,
$p = 1$ (linear),
$p = 2$ (quadratic).




### Characteristics

+ Easily applicable in practice
+ Provides rich information
– May be long
▶ `insapack.chirp`

# Experiment design: choice of the exciting signal

## Chirp signals (`insapack.chirp`)

```
% Description
% Computes a chirp signal sweeping over FBND frequency range.
%
% Syntax
%  [uc,tc,info] = insapack.chirp(Ns,Ts,FBND,REV,TYPE,SHOW)
%
% Input arguments
% - Ns   : number of samples (integer)
% - Ts   : sampling period [s] (>0 real)
% - FBND : frequency range [Hz] (2x1 real vector)
% - REV  : reverse signal (boolean)
%          - false: fmin to max
%          - true: fmax to fmin
% - TYPE : variation of the chirp (string)
%          - 'linear'
%          - 'quadratic'
%          - 'logarithmic'
% - SHOW : plot signal (boolean)
%
% Output arguments
% - uc   : chirp signal in [-1,1]
% - tc   : time samples
% - info : additional information
```

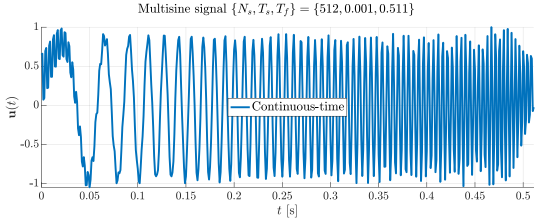
# Experiment design: choice of the exciting signal

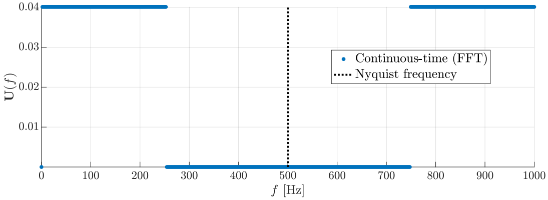
## Multi-sine signals

$$\mathbf{u}(t) = \sum_{k=1}^{F} A_k \cos(2\pi k f_0 t + \phi_k)$$

$$\phi_k = -\frac{k(k-1)\pi}{F} \text{ (Schroder)}$$

$\phi_k \in [0, 2\pi($ (random).




### Characteristics

- + Easily applicable in practice
- + Provides very rich information
- + Can be adapted to deal with non-linearities
- ▶ `insapack.multisine`

# Experiment design: choice of the exciting signal

## Multi-sine signals (`insapack.multisine`)

```
% Description
% Computes a multi-sine signal sweeping over FBND frequency range.
%
% Syntax
%  [uc,tc,info] = insapack.multisine(Ns,Ts,FBND,RPHI,ODD,REV,SHOW)
%
% Input arguments
% - Ns   : number of samples (integer)
% - Ts   : sampling period [s] (>0 real)
% - FBND : frequency range [Hz] (2x1 real vector)
% - RPHI : random phase (boolean)
%          - false: Schroeder multisine
%          - true: random multisine
% - ODD  : frequency excited (used for nonlinear detection)
%          - 'all': all frequencies excited
%          - 'odd': odd frequencies excited (even discarded)
%          - 'odd-odd': odd-odd frequencies excited
%          - 'odd-rnd': odd frequencies excited (random block not)
% - REV  : reverse signal (boolean)
%          - false: fmin to max
%          - true: fmax to fmin
% - SHOW : plot signal (boolean)
%
% Output arguments
% - uc   : chirp signal in [-1,1]
% - tc   : time samples
% - info : additional information
```

# Content

# Some guidelines

## Pre- and post-treatment

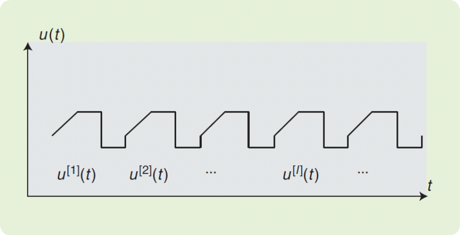
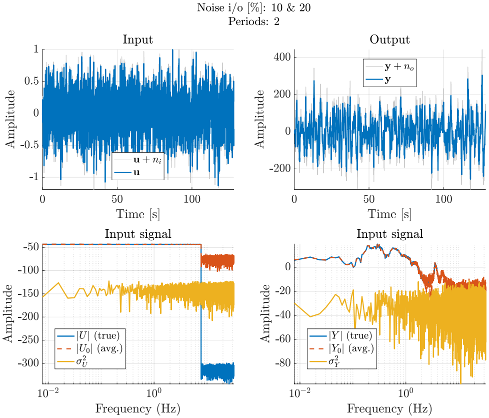



**FIGURE 4** Calculating the sample mean and sample variance of a periodic signal starting from multiple measured periods. The input $u(t)$ and output $y(t)$ are measured over multiple periods and broken into subrecords of one period each, for example $u^{[l]}, l = 1, \dots, P$. The fast Fourier transform is applied to each subrecord. In the next step, the mean value and the (co)variance are calculated as a function of the frequency.

- Analyse the setup
- Verify anti-aliasing filters exist and have proper cut-off frequency
- Verify data synchronization
  If not synchronized, re-sampling may be applied
- Pre-processing: mean and trend removal, missing data, pre-filtering
- Store reference/control/output signals
- Use data for "learning" and "validation"
- Use as much as possible periodic signal excitations
- **Average data (to account for noise)**
- Ask experimental team !

# Some guidelines

## Non-parametric analysis (over $P$ experiments)




### Signals variance

$$\mathbf{U}_0 = \frac{1}{P}\sum \mathbf{U}$$

$$\mathbf{Y}_0 = \frac{1}{P}\sum \mathbf{Y}$$

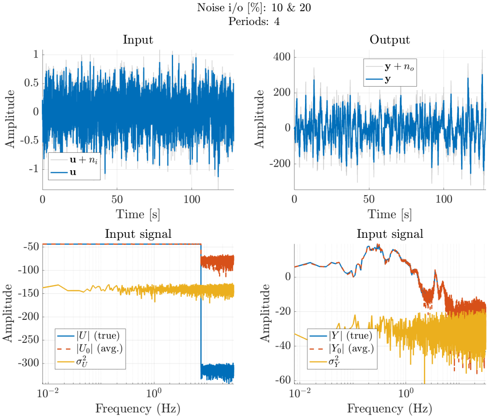
$$\sigma^2_{\mathbf{U}} = \frac{1}{P-1}\sum |\mathbf{U}-\mathbf{U}_0|^2$$

$$\sigma^2_{\mathbf{Y}} = \frac{1}{P-1}\sum |\mathbf{Y}-\mathbf{Y}_0|^2$$

$$\sigma^2_{\mathbf{UY}} = \frac{1}{P-1}\sum (\mathbf{Y}-\mathbf{Y}_0)\overline{(\mathbf{U}-\mathbf{U}_0)}$$

# Some guidelines

## Non-parametric analysis (over $P$ experiments)




### Signals variance

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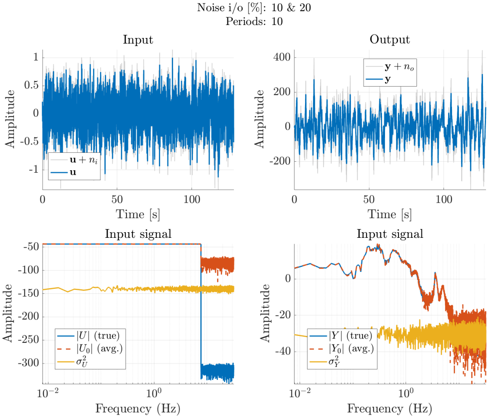
# Some guidelines

## Non-parametric analysis (over $P$ experiments)




### Signals variance

$$\mathbf{U}_0 = \frac{1}{P}\sum \mathbf{U}$$

$$\mathbf{Y}_0 = \frac{1}{P}\sum \mathbf{Y}$$

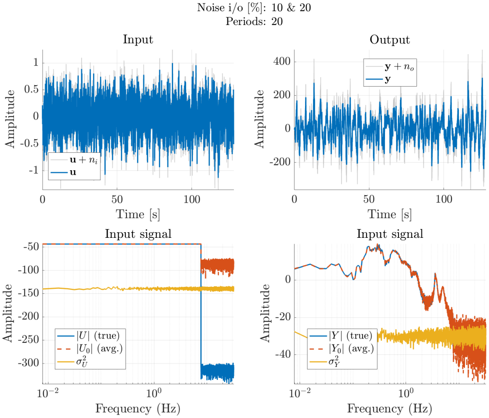
$$\sigma_{\mathbf{U}}^2 = \frac{1}{P-1}\sum |\mathbf{U}-\mathbf{U}_0|^2$$

$$\sigma_{\mathbf{Y}}^2 = \frac{1}{P-1}\sum |\mathbf{Y}-\mathbf{Y}_0|^2$$

$$\sigma_{\mathbf{UY}}^2 = \frac{1}{P-1}\sum (\mathbf{Y}-\mathbf{Y}_0)\overline{(\mathbf{U}-\mathbf{U}_0)}$$

# Some guidelines

## Non-parametric analysis (over $P$ experiments)

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$$\mathbf{Y}_0 = \frac{1}{P}\sum \mathbf{Y}$$

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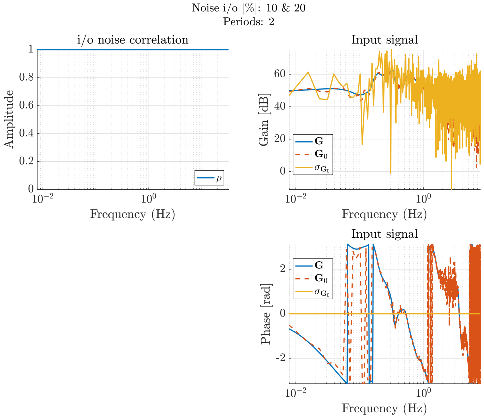
$$\sigma^2_{\mathbf{UY}} = \frac{1}{P-1}\sum (\mathbf{Y}-\mathbf{Y}_0)\overline{(\mathbf{U}-\mathbf{U}_0)}$$

# Some guidelines

## Non-parametric analysis (over $P$ experiments)




### I/O noise correlation

$$\rho = \frac{\sigma^2_{\mathbf{UY}}}{\sigma_{\mathbf{U}}\sigma_{\mathbf{Y}}}$$

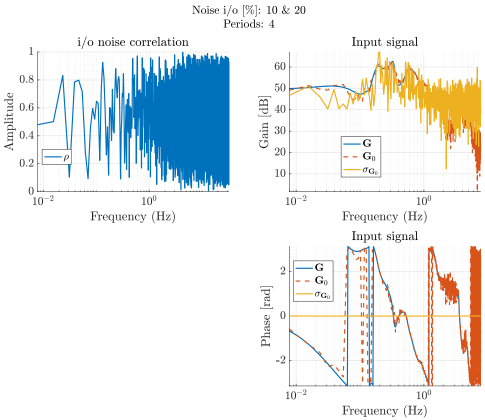
### Model variance

The variability of $\mathbf{G}(k)$ around its expected value is characterized by the variance $\sigma^2_{\mathbf{G}}$. The larger, the wider the spread around the expected value. For a Gaussian pdf, the interval: $[-1.96\sigma_{\mathbf{G}}; +1.96\sigma_{\mathbf{G}}]$ corresponds to the 95% confidence interval.

$$\sigma^2_{\mathbf{G}} = \frac{1}{P}|\mathbf{G}_0|^2 \left( \frac{\sigma^2_{\mathbf{Y}}}{|\mathbf{Y}_0|^2} + \frac{\sigma^2_{\mathbf{U}}}{|\mathbf{U}_0|^2} - 2\text{Re}\left( \frac{\sigma^2_{\mathbf{UY}}}{\mathbf{Y}_0\overline{\mathbf{U}_0}} \right) \right)$$

# Some guidelines

## Non-parametric analysis (over $P$ experiments)

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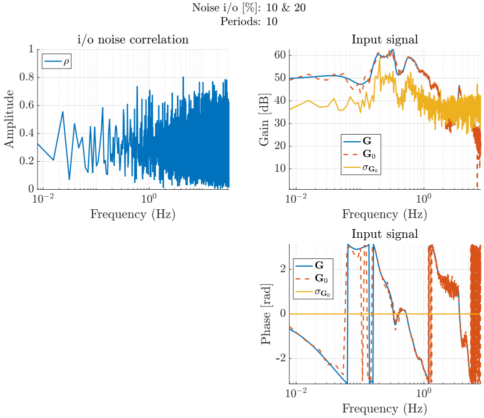
# Some guidelines

## Non-parametric analysis (over $P$ experiments)

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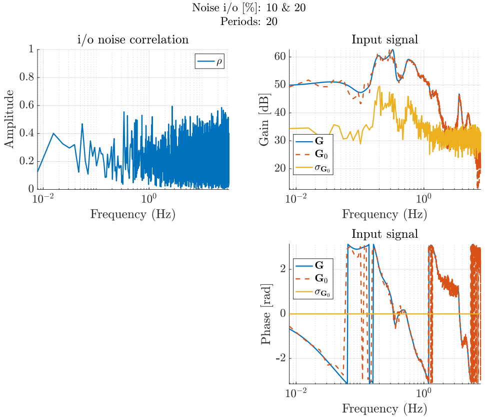
# Some guidelines

## Non-parametric analysis (over $P$ experiments)

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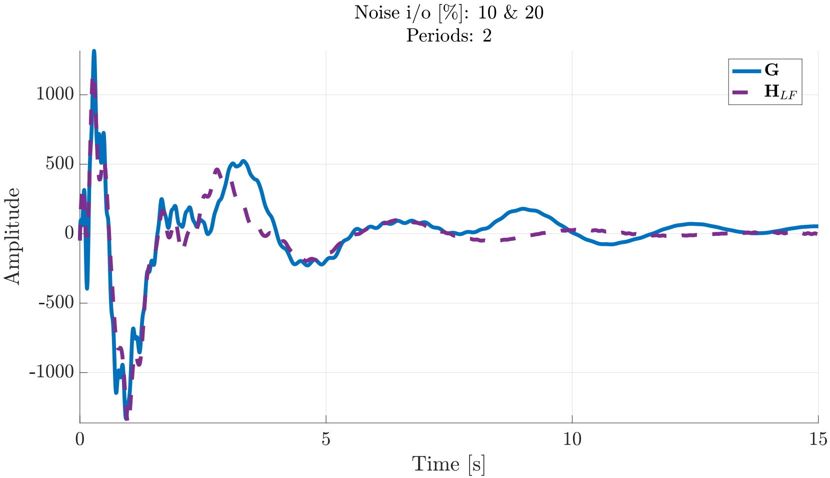
$$\sigma^2_{\mathbf{G}} = \frac{1}{P}|\mathbf{G}_0|^2 \left( \frac{\sigma^2_{\mathbf{Y}}}{|\mathbf{Y}_0|^2} + \frac{\sigma^2_{\mathbf{U}}}{|\mathbf{U}_0|^2} - 2\text{Re}\left( \frac{\sigma^2_{\mathbf{UY}}}{\mathbf{Y}_0\overline{\mathbf{U}_0}} \right) \right)$$

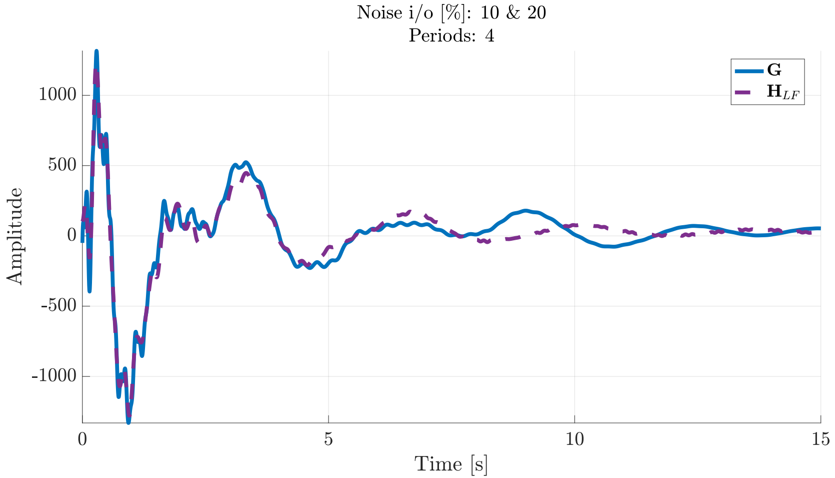
# Some guidelines

## Non-parametric analysis (over $P$ experiments) - glimpse of identification ($r = 30$)



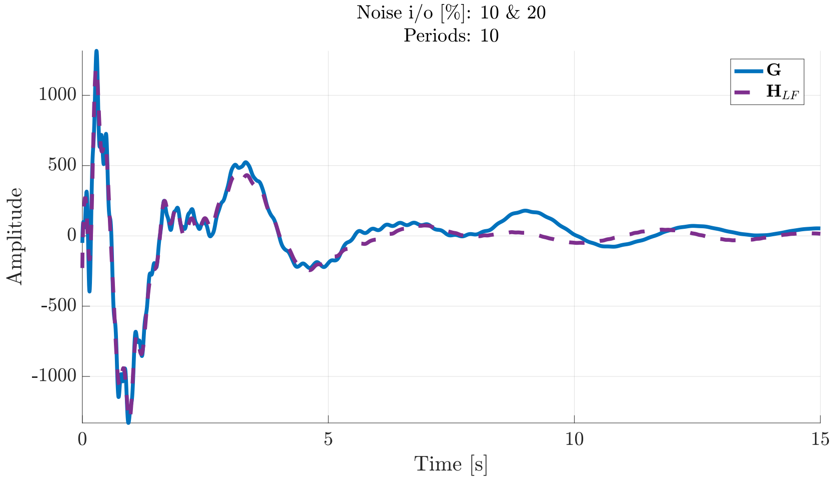
# Some guidelines

## Non-parametric analysis (over $P$ experiments) - glimpse of identification ($r = 30$)



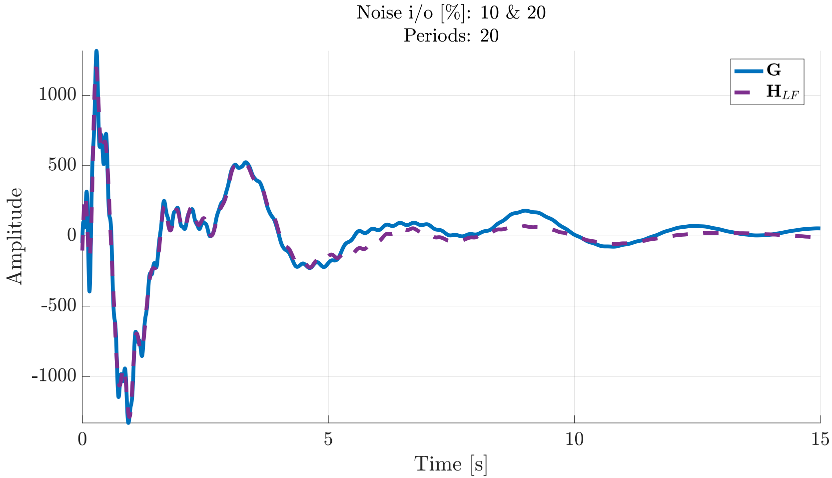
# Some guidelines

## Non-parametric analysis (over $P$ experiments) - glimpse of identification ($r = 30$)



# Some guidelines

## Non-parametric analysis (over $P$ experiments) - glimpse of identification ($r = 30$)



# Some guidelines

## The closed-loop case

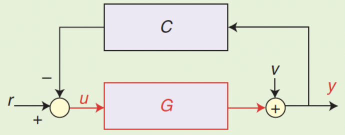



**FIGURE S8** The frequency response function (FRF) measurement under closed-loop conditions requires special care. The FRF of the system $G$ captured in a feedback loop is measured starting from the measured input $u(t)$ and output $y(t)$. This leads to a bias because the input $u$ is correlated with the noise $v$ through the feedback path $C$.

- Output $\mathbf{y}(t)$ depends on both measured input $\mathbf{u}(t)$ and disturbance source $\mathbf{v}(t)$
- The FRF converges to

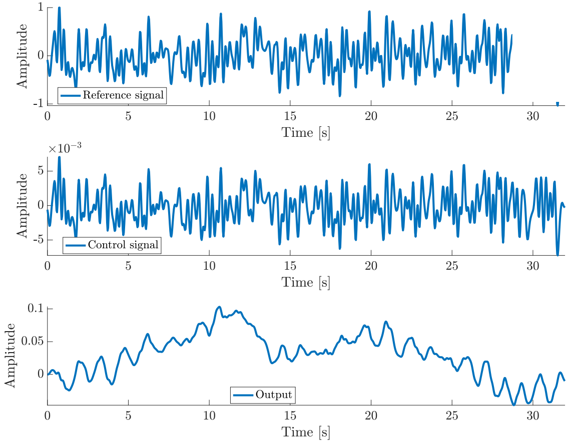
$$\hat{\mathbf{G}} = \frac{\mathbf{G}S_{\mathbf{rr}} - \mathbf{C}S_{\mathbf{yy}}}{S_{\mathbf{rr}} + |\mathbf{C}|^2 S_{\mathbf{vv}}}$$

- $\hat{\mathbf{G}} = \mathbf{G}$, $\mathbf{r}$ dominates over $\mathbf{v}$ ($S_{\mathbf{vv}} = 0$)
- $\hat{\mathbf{G}} = \frac{-1}{\mathbf{C}}$, $\mathbf{v}$ dominates over $\mathbf{r}$ ($S_{\mathbf{rr}} = 0$)
- If $\mathbf{r}$ is exactly known,

$$\hat{\mathbf{G}} = \frac{\mathbf{G}_{\mathbf{yr}}}{\mathbf{G}_{\mathbf{ur}}} = \frac{S_{\mathbf{yr}}}{S_{\mathbf{ur}}}$$

# Some guidelines

## The closed-loop case



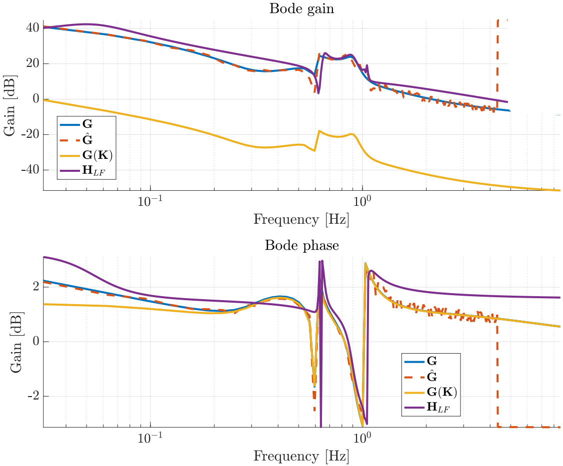

### Example

**G** dimension 50

**PI** controller

# Some guidelines

## The closed-loop case




### Example

**G** dimension 50

**PI** controller

**Demo #1 (+insapack.mlbs)**

**Demo #2 (+insapack.multisine)**

# Linear dynamical system identification

... basic elements and Labs guidelines

Charles Poussot-Vassal
February 20, 2026

INSTITUT NATIONAL DES SCIENCES APPLIQUÉES TOULOUSE