The Loewner framework

A toolkit for identification, approximation, realization, projection...

C. Poussot-Vassal
November 28, 2026

Forewords

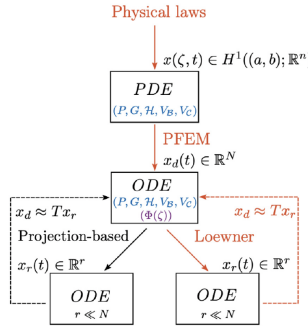
Loewner Framework landmark

Main discussion

- Rational interpolation;
  Rational interpolation without computation [New];
- Data-driven realization and identification (LTI);
  Connection with the Petrov-Galerkin framework;
  And port-Hamiltoninan structure.

Side discussions

- PDEs applications
- pH applications
- experimental applications
- extensions to parametric



From physics to pH-ROM
(by J.P. Toledo-Zucco)

Data (raw)

$$P := \{(x_l, \mathbf{y}_l) \mid l = 1, \cdots, N\} \quad x_l \text{ distinct}$$

Data ("any" x_l, $\mathbf{y}_l$ split)

$$P := \left\{ \begin{array}{lcll} P_c & := & \{(\lambda_j, \mathbf{w}_j)\ j = 1, \ldots, k\} & k \text{ columns} \\ P_r & := & \{(\mu_i, \mathbf{v}_i)\ i = 1, \ldots, q\} & q \text{ rows} \end{array} \right. \qquad N = k + q$$

Rational form (Lagrangian basis)

$$\mathbf{H}(s) = \frac{\sum_{j=1}^{k} \frac{\alpha_j \mathbf{w}_j}{s - \lambda_j}}{\sum_{j=1}^{k} \frac{\alpha_j}{s - \lambda_j}}$$

Realization form (Lagrangian basis)

$$\mathbf{H}(s) = C(sE - A)^{-1}B$$

Loewner

- A.C. Antoulas and B.D.O Anderson "On the scalar rational interpolation problem". IMA J. Math. Control. Inf. 3(2-3), 61-88 (1986).
- A.J. Mayo and A.C. Antoulas, "A framework for the solution of the generalized realization problem", Linear Algebra and its Applications, vol. 425(2-3), (2007).
- A.C. Antoulas, S. Lefteriu and A.C. Ionita, "A tutorial introduction to the Loewner framework for model reduction". In: Model Reduction and Approximation, chap. 8, pp. 335-376. SIAM (2017).

Loewner & pH

- A.C. Antoulas, "A new result on passivity preserving model reduction", Systems & Control Letters, vol. 54, (2005).
- P. Benner, P. Goyal and P. Van Dooren, "Identification of port-Hamiltonian systems from frequency response data", Systems & Control Letters, Vol. 143, p. 104741 (2020).

Content

Forewords

Loewner rational approximation

Loewner realization

Conclusion

Loewner rational approximation

Scalar rational interpolation and the Loewner matrix (a rational Lagrange-type formula)

Given

$$P := \{(x_l, \mathbf{y}_l) \mid l = 1, \cdots, N\} \quad x_l \text{ distinct}$$

the idea behind is to use a formula for rational interpolants similar to the Lagrange polynomial. First we partition the array $\mathcal{D}$ in two disjoint subarrays:

$$\begin{array}{rcl} P_c & := & \{(\lambda_j, \mathbf{w}_j)\ j = 1, \ldots, k\} \\ P_r & := & \{(\mu_i, \mathbf{v}_i)\ i = 1, \ldots, q\} \end{array}$$

e.g. $N = k + q$

$$\begin{array}{rclcrcll} \lambda_j & = & x_j & & \mathbf{w}_j & = & \mathbf{y}_j & j = 1, \ldots, k \\ \mu_i & = & x_{k+i} & & \mathbf{v}_i & = & \mathbf{y}_{k+i} & i = 1, \ldots, q \end{array}$$

Using array P_c, for constants α_j and $\mathbf{w}_j$, consider $\mathbf{H}(s)$ given by

$$\sum_{j=1}^{k} \alpha_j \frac{\mathbf{H}(s) - \mathbf{w}_j}{s - \lambda_j} = 0 \quad \text{for} \quad \alpha_j \neq 0$$

A.C. Antoulas and B.D.O Anderson, *"On the scalar rational interpolation problem"*, IMA Journal of Mathematical Control and Information, vol. 3(2-3), 1986.

J.-P. Berrut and L. N. Trefethen, *"Barycentric Lagrange interpolation"*, SIAM Review, vol. 46, 2004.

Loewner rational approximation

Scalar rational interpolation and the Loewner matrix (a rational Lagrange-type formula)

P_c Column interpolation points (constraints by construction)

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j$$

guaranteed by

$$\mathbf{H}(s) = \frac{\sum_{j=1}^{k} \frac{\alpha_j \mathbf{w}_j}{s - \lambda_j}}{\sum_{j=1}^{k} \frac{\alpha_j}{s - \lambda_j}}$$

P_r Row interpolation points (constraints by linear algebra)

$$\mathbf{H}(\mu_i) = \mathbf{v}_i$$

guaranteed by

$$\mathbb{L}\mathbf{c} = 0$$

where

$$\mathbb{L} = \begin{bmatrix} \frac{\mathbf{v}_1 - \mathbf{w}_1}{\mu_1 - \lambda_1} & \cdots & \frac{\mathbf{v}_1 - \mathbf{w}_k}{\mu_1 - \lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_q - \mathbf{w}_1}{\mu_q - \lambda_1} & \cdots & \frac{\mathbf{v}_q - \mathbf{w}_k}{\mu_q - \lambda_k} \end{bmatrix} \in \mathbb{C}^{q \times k} \text{ and } \mathbf{c} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{bmatrix} \in \mathbb{C}^k.$$

Loewner rational approximation

Scalar rational interpolation and the Loewner matrix (a rational Lagrange-type formula)

P_c Column interpolation points (constraints by construction)

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j$$

guaranteed by

$$\mathbf{H}(s) = \frac{\sum_{j=1}^{k} \frac{\alpha_j \mathbf{w}_j}{s-\lambda_j}}{\sum_{j=1}^{k} \frac{\alpha_j}{s-\lambda_j}}$$

P_r Row interpolation points (constraints by linear algebra)

$$\mathbf{H}(\mu_i) = \mathbf{v}_i$$

guaranteed by

$$\mathbb{L}\mathbf{c} = 0$$

where

$$\mathbb{L} = \begin{bmatrix} \frac{\mathbf{v}_1 - \mathbf{w}_1}{\mu_1 - \lambda_1} & \cdots & \frac{\mathbf{v}_1 - \mathbf{w}_k}{\mu_1 - \lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_q - \mathbf{w}_1}{\mu_q - \lambda_1} & \cdots & \frac{\mathbf{v}_q - \mathbf{w}_k}{\mu_q - \lambda_k} \end{bmatrix} \in \mathbb{C}^{q \times k} \text{ and } \mathbf{c} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{bmatrix} \in \mathbb{C}^k.$$

Loewner rational approximation

Rank and McMillan degree

Lemma

Given the rational function $\mathbf{H}$ and an array of points P, where $\mathbf{y}_j = \mathbf{H}(x_j)$ and x_j is not a pole of $\mathbf{H}$, let $\mathbb{L}$ be a $q \times k$ Loewner matrix, for some partitioning P_c, P_r of P. Then

$$q, k \geq \mathbf{deg}(\mathbf{H}) \Rightarrow \mathbf{rank}\ \mathbb{L} = \mathbf{deg}(\mathbf{H})$$

It follows that every square sub-Loewner matrix of size $\deg(\mathbf{H})$, is nonsingular.

Proof: let (E, A, B, C) be a minimal descriptor realization of $\mathbf{H}$, of order n then,

$$\begin{aligned}
[\mathbb{L}]_{i,j} = \frac{\mathbf{v}_i - \mathbf{w}_j}{\mu_i - \lambda_j} = \frac{\mathbf{H}(\mu_i) - \mathbf{H}(\lambda_j)}{\mu_i - \lambda_j} &= \frac{C(\mu_i E - A)^{-1} - (\lambda_j E - A)^{-1} B}{\mu_i - \lambda_j} \\
&= \frac{C(\mu_i E - A)^{-1}\Big((\lambda_j E - A) - (\mu_i E - A)\Big)(\lambda_j E - A)^{-1} B}{\mu_i - \lambda_j} \\
&= -C(\mu_i E - A)^{-1} E (\lambda_j E - A)^{-1} B \\
&= -[\mathcal{O}_q]_i E [\mathcal{R}_k]_j
\end{aligned}$$

A.C. Antoulas and B.D.O Anderson, *"On the scalar rational interpolation problem"*, IMA Journal of Mathematical Control and Information, vol. 3(2-3), 1986.

Loewner rational approximation

Rank and McMillan degree

Lemma

Given the rational function $\mathbf{H}$ and an array of points P, where $\mathbf{y}_j = \mathbf{H}(x_j)$ and x_j is not a pole of $\mathbf{H}$, let $\mathbb{L}$ be a $q \times k$ Loewner matrix, for some partitioning P_c, P_r of P. Then

$$q, k \geq \mathbf{deg}(\mathbf{H}) \Rightarrow \mathbf{rank}\ \mathbb{L} = \mathbf{deg}(\mathbf{H})$$

It follows that every square sub-Loewner matrix of size $\deg(\mathbf{H})$, is nonsingular.

Proof: let (E, A, B, C) be a minimal descriptor realization of $\mathbf{H}$, of order n then,

$$\begin{aligned}
[\mathbb{L}]_{i,j} = \frac{\mathbf{v}_i - \mathbf{w}_j}{\mu_i - \lambda_j} = \frac{\mathbf{H}(\mu_i) - \mathbf{H}(\lambda_j)}{\mu_i - \lambda_j} &= \frac{C(\mu_i E - A)^{-1} - (\lambda_j E - A)^{-1} B}{\mu_i - \lambda_j} \\
&= \frac{C(\mu_i E - A)^{-1}\Big((\lambda_j E - A) - (\mu_i E - A)\Big)(\lambda_j E - A)^{-1} B}{\mu_i - \lambda_j} \\
&= -C(\mu_i E - A)^{-1} E (\lambda_j E - A)^{-1} B \\
&= -[\mathcal{O}_q]_i E [\mathcal{R}_k]_j
\end{aligned}$$

A.C. Antoulas and B.D.O Anderson, *"On the scalar rational interpolation problem"*, IMA Journal of Mathematical Control and Information, vol. 3(2-3), 1986.

Loewner rational approximation

Rank and McMillan degree

$$[\mathbb{L}]_{i,j} = -[\mathcal{O}_q]_i E [\mathcal{R}_k]_j$$

where the generalized observability and reachability matrices are,

$$\begin{array}{rcll} \mathcal{O}_q & = & \begin{bmatrix} (\mu_1 E - A)^{-\top} C^\top, \cdots, (\mu_q E - A)^{-\top} C^\top \end{bmatrix}^\top & \in \mathbb{C}^{q \times n} \\ \mathcal{R}_k & = & \begin{bmatrix} (\lambda_1 E - A)^{-1} B, \cdots, (\lambda_k E - A)^{-1} B \end{bmatrix} & \in \mathbb{C}^{n \times k} \end{array}$$

Note that if (E, A, B, C) is controllable and observable, and $k, q \geq n$, then $\mathbf{rank}\ (\mathcal{O}_q) = n$ and $\mathbf{rank}\ (\mathcal{R}_k) = n$

Similarity with Hankel

Considering $\lambda_j, \mu_i = \infty$ leads to the classical Kalman observability and reachability matrices

$$\begin{array}{rcll} \mathcal{O}_q & = & \begin{bmatrix} C^\top, A^\top C^\top, \cdots, (A^q)^\top C^\top \end{bmatrix}^\top & \in \mathbb{C}^{q \times n} \\ \mathcal{R}_k & = & \begin{bmatrix} B, \cdots, A^k B \end{bmatrix} & \in \mathbb{C}^{n \times k} \\ \mathcal{H} & = & \mathcal{O}_q \mathcal{R}_k & \in \mathbb{C}^{q \times k} \text{ (matrix of Markov parameters)} \end{array}$$

Loewner rational approximation

Rank and McMillan degree

$$[\mathbb{L}]_{i,j} = -[\mathcal{O}_q]_i E [\mathcal{R}_k]_j$$

where the generalized observability and reachability matrices are,

$$\begin{aligned} \mathcal{O}_q &= \begin{bmatrix} (\mu_1 E - A)^{-\top} C^\top, \cdots, (\mu_q E - A)^{-\top} C^\top \end{bmatrix}^\top && \in \mathbb{C}^{q \times n} \\ \mathcal{R}_k &= \begin{bmatrix} (\lambda_1 E - A)^{-1} B, \cdots, (\lambda_k E - A)^{-1} B \end{bmatrix} && \in \mathbb{C}^{n \times k} \end{aligned}$$

Note that if (E, A, B, C) is controllable and observable, and $k, q \geq n$, then $\mathbf{rank}\ (\mathcal{O}_q) = n$ and $\mathbf{rank}\ (\mathcal{R}_k) = n$

Similarity with Hankel

Considering $\lambda_j, \mu_i = \infty$ leads to the classical Kalman observability and reachability matrices

$$\begin{aligned} \mathcal{O}_q &= \begin{bmatrix} C^\top, A^\top C^\top, \cdots, (A^q)^\top C^\top \end{bmatrix}^\top && \in \mathbb{C}^{q \times n} \\ \mathcal{R}_k &= \begin{bmatrix} B, \cdots, A^k B \end{bmatrix} && \in \mathbb{C}^{n \times k} \\ \mathcal{H} &= \mathcal{O}_q \mathcal{R}_k && \in \mathbb{C}^{q \times k} \text{ (matrix of Markov parameters)} \end{aligned}$$

Loewner rational approximation

Edifying example #1

Data generated from $\mathbf{G}(s) = \dfrac{s^2+4}{s+1}$ of complexity (2), with $k = 3$, $q = 4$

$$\left.\begin{array}{rcl} \lambda_j & = & \begin{bmatrix} 1 & 3 & 5 \end{bmatrix} \\ \mu_i & = & \begin{bmatrix} 0 & 2 & 4 & 6 \end{bmatrix} \end{array}\right\} \xrightarrow{\mathbf{H}} \left\{\begin{array}{rcl} \mathbf{w}_j & = & \begin{bmatrix} \frac{5}{2} & \frac{13}{4} & \frac{29}{6} \end{bmatrix} \\ \mathbf{v}_i & = & \begin{bmatrix} 4 & \frac{8}{3} & 4 & \frac{40}{7} \end{bmatrix} \end{array}\right.$$

$$\mathbb{L} = \begin{bmatrix} -\frac{3}{2} & -\frac{1}{4} & \frac{1}{6} \\ \frac{1}{6} & \frac{7}{12} & \frac{13}{18} \\ \frac{1}{2} & \frac{3}{4} & \frac{5}{6} \\ \frac{9}{14} & \frac{23}{28} & \frac{37}{42} \end{bmatrix} \qquad \mathbf{c} = \begin{bmatrix} \frac{1}{3} \\ -\frac{4}{3} \\ 1 \end{bmatrix}$$

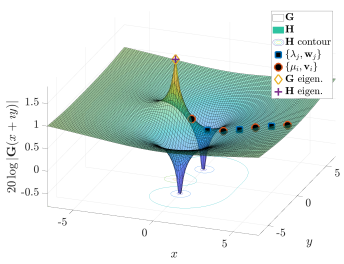
$$\mathbf{H}(s) = \frac{\frac{5}{6\,(s-1)} - \frac{13}{3\,(s-3)} + \frac{29}{6\,(s-5)}}{\frac{1}{3\,(s-1)} - \frac{4}{3\,(s-3)} + \frac{1}{s-5}} = \frac{s^2+4}{s+1}$$

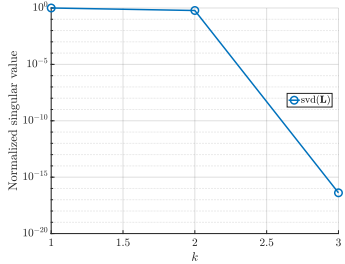
Loewner rational approximation

Edifying example #1

Data generated from $\mathbf{G}(s) = \dfrac{s^2+4}{s+1}$ of complexity (2), with $k=3$, $q=4$

$$\left.\begin{array}{rcl} \lambda_j & = & \begin{bmatrix} 1 & 3 & 5 \end{bmatrix} \\ \mu_i & = & \begin{bmatrix} 0 & 2 & 4 & 6 \end{bmatrix} \end{array}\right\} \xrightarrow{\mathbf{H}} \left\{\begin{array}{rcl} \mathbf{w}_j & = & \begin{bmatrix} \frac{5}{2} & \frac{13}{4} & \frac{29}{6} \end{bmatrix} \\ \mathbf{v}_i & = & \begin{bmatrix} 4 & \frac{8}{3} & 4 & \frac{40}{7} \end{bmatrix} \end{array}\right.$$





Loewner rational approximation

Edifying example #2

Data generated from $\mathbf{G}(s) = \dfrac{1}{s^2 + s\ (1+1\mathrm{i}) + 1}$ of complexity (2), with $k = 3$, $q = 3$

$$\left.\begin{array}{rcl} \lambda_j & = & \begin{bmatrix} 1 & -2+1\mathrm{i} & 1\mathrm{i} \end{bmatrix} \\ \mu_i & = & \begin{bmatrix} 0 & 2 & -1 \end{bmatrix} \end{array}\right\} \xrightarrow{\mathbf{H}} \left\{\begin{array}{rcl} \mathbf{w}_j & = & \begin{bmatrix} 0.3-0.1\mathrm{i} & 0.038+0.19\mathrm{i} & -0.5-0.5\mathrm{i} \end{bmatrix} \\ \mathbf{v}_i & = & \begin{bmatrix} 1.0 & 0.13-0.038\mathrm{i} & 0.5+0.5\mathrm{i} \end{bmatrix} \end{array}\right.$$

$$\mathbb{L} = \begin{bmatrix} -0.7-0.1\mathrm{i} & 0.42+0.12\mathrm{i} & -0.5+1.5\mathrm{i} \\ -0.17+0.062\mathrm{i} & 0.036-0.049\mathrm{i} & 0.16+0.31\mathrm{i} \\ -0.1-0.3\mathrm{i} & 0.077+0.38\mathrm{i} & -1.0 \end{bmatrix} \qquad \mathbf{c} = \begin{bmatrix} \frac{1}{5}+\frac{7}{5}\mathrm{i} \\ \frac{4}{5}-\frac{7}{5}\mathrm{i} \\ 1 \end{bmatrix}$$

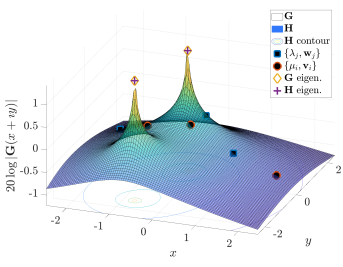
$$\mathbf{H}(s) = \frac{\frac{0.2+0.4\mathrm{i}}{s-1.0} + \frac{-0.5-0.5\mathrm{i}}{s-1.0\mathrm{i}} + \frac{0.3+0.1\mathrm{i}}{s+2.0-1.0\mathrm{i}}}{\frac{0.2+1.4\mathrm{i}}{s-1.0} + \frac{1}{s-1.0\mathrm{i}} + \frac{0.8-1.4\mathrm{i}}{s+2.0-1.0\mathrm{i}}} = \frac{1.0}{s^2 + s\ (1.0+1.0\mathrm{i}) + 1.0}$$

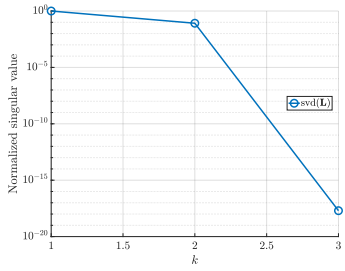
Loewner rational approximation

Edifying example #2

Data generated from $\mathbf{G}(s) = \dfrac{1}{s^2 + s\,(1+1\mathrm{i}) + 1}$ of complexity (2), with $k = 3$, $q = 3$

$$\left.\begin{array}{rcl} \lambda_j & = & \begin{bmatrix} 1 & -2+1\mathrm{i} & 1\mathrm{i} \end{bmatrix} \\ \mu_i & = & \begin{bmatrix} 0 & 2 & -1 \end{bmatrix} \end{array}\right\} \xrightarrow{\mathbf{H}} \left\{\begin{array}{rcl} \mathbf{w}_j & = & \begin{bmatrix} 0.3-0.1\mathrm{i} & 0.038+0.19\mathrm{i} & -0.5-0.5\mathrm{i} \end{bmatrix} \\ \mathbf{v}_i & = & \begin{bmatrix} 1.0 & 0.13-0.038\mathrm{i} & 0.5+0.5\mathrm{i} \end{bmatrix} \end{array}\right.$$

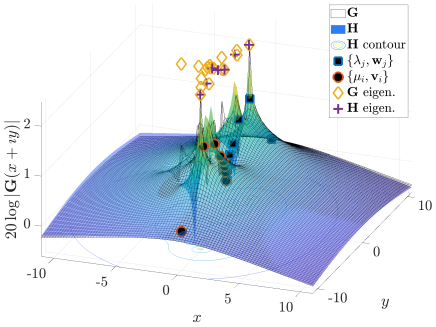


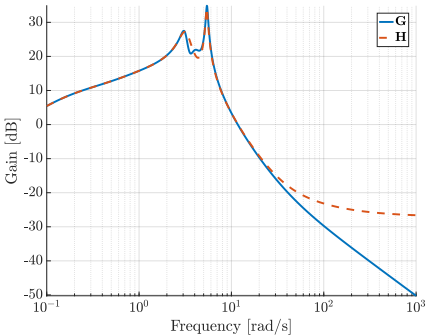


Loewner rational approximation

Edifying example #3

Data generated from a random model $\mathbf{G}(s)$ of complexity (19), with $k = 8$, $q = 8$





Loewner rational approximation

Summary

$$\begin{cases} P_c & := & \{(\lambda_j; \mathbf{w}_j)\,,\ j = 1, \ldots, k\} \\ P_r & := & \{(\mu_i; \mathbf{v}_i)\,,\ i = 1, \ldots, q\} \end{cases}$$

Loewner matrix

$$\mathbb{L} \in \mathbb{C}^{q \times k}$$

$$(\mathbb{L})_{i,j} = \frac{\mathbf{v}_i - \mathbf{w}_j}{\mu_i - \lambda_j}$$

Lagrangian form

$$\mathbf{H}(s) = \frac{\sum_{j=1}^{k} \frac{\alpha_j \mathbf{w}_j}{s - \lambda_j}}{\sum_{j=1}^{k} \frac{\alpha_j}{s - \lambda_j}} \approx \frac{\sum_{j=1}^{r} \frac{\hat{\alpha}_j \mathbf{w}_j}{s - \lambda_j}}{\sum_{j=1}^{r} \frac{\hat{\alpha}_j}{s - \lambda_j}} = \hat{\mathbf{H}}(s) \quad r < k$$

Null space

$$\mathbf{c} = \mathbf{span}(\mathcal{N}(\mathbb{L})) \approx \hat{\mathbf{c}}$$

$$\mathbf{c} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{bmatrix} \in \mathbb{C}^k$$

$$\hat{\mathbf{c}} = \begin{bmatrix} \hat{\alpha}_1 \\ \hat{\alpha}_2 \\ \vdots \\ \hat{\alpha}_r \end{bmatrix} \in \mathbb{C}^r$$

Loewner rational approximation

Summary . . . and extension

In the 1-variable case

- the q-left and k-right function samples;
- the associated Loewner matrix $\mathbb{L} \in \mathbb{C}^{q \times k}$;
- a (right) null-space or approximate null-space vector $\hat{\mathbf{c}} \in \mathbb{C}^k$;
- the barycentric formula, which provides an (approximate) interpolant of the data;
- the computational/storage complexity involved is k^3 `flop`/ k^2 MB

In the n-variables case

- the q_l-left and k_l-right function samples ($l = 1, \cdots, n$);
- the associated Loewner matrix $\mathbb{L} \in \mathbb{C}^{Q \times K}$ ($Q = q_1 q_2 \cdots q_n$, $K = k_1 k_2 \cdots k_n$);
- a (right) null-space or approximate null-space vector $\hat{\mathbf{c}} \in \mathbb{C}^K$;
- the barycentric formula, which provides an (approximate) interpolant of the data;
- the computational/storage complexity involved is K^3 `flop`/ K^2 MB
  $\rightarrow k_1^3 + k_1 k_2^3 + k_1 k_2 k_3^3 + \cdots + k_1 k_2 \cdots k_{n-1} k_n^3$
  $\rightarrow \max_{l=1,\cdots,n} k_l^2$

A. C. Antoulas, I. V. Gosea and C. P-V., *"On the Loewner framework, the Kolmogorov superposition theorem, and the curse of dimensionality"*, in SIAM Review, Vol. 67(4), 2025.

Loewner rational approximation

Summary . . . and extension

In the 1-variable case

- the q-left and k-right function samples;
- the associated Loewner matrix $\mathbb{L} \in \mathbb{C}^{q \times k}$;
- a (right) null-space or approximate null-space vector $\hat{\mathbf{c}} \in \mathbb{C}^k$;
- the barycentric formula, which provides an (approximate) interpolant of the data;
- the computational/storage complexity involved is k^3 `flop`/ k^2 MB

In the n-variables case

- the q_l-left and k_l-right function samples ($l = 1, \cdots, n$);
- the associated Loewner matrix $\mathbb{L} \in \mathbb{C}^{Q \times K}$ ($Q = q_1 q_2 \cdots q_n$, $K = k_1 k_2 \cdots k_n$);
- a (right) null-space or approximate null-space vector $\hat{\mathbf{c}} \in \mathbb{C}^K$;
- the barycentric formula, which provides an (approximate) interpolant of the data;
- the computational/storage complexity involved is K^3 `flop`/ K^2 MB
  $\rightarrow k_1^3 + k_1 k_2^3 + k_1 k_2 k_3^3 + \cdots + k_1 k_2 \cdots k_{n-1} k_n^3$
  $\rightarrow \max_{l=1,\cdots,n} k_l^2$

A. C. Antoulas, I. V. Gosea and C. P-V., *"On the Loewner framework, the Kolmogorov superposition theorem, and the curse of dimensionality"*, in SIAM Review, Vol. 67(4), 2025.

Loewner rational approximation

Lagrangian interpolation without computation (slice of the result)

The barycentric weight of the **denominator** determine the null-space of an associated Loewner matrix

$$\mathbf{H} = \frac{\mathbf{n}(s)}{\mathbf{d}(s)} \quad \text{of degree } k-1$$

Theorem

$$\begin{aligned} \mathbf{Bary}^{\text{den}} &= \begin{bmatrix} \gamma_1 \mathbf{d}(\lambda_1) & \cdots & \gamma_k \mathbf{d}(\lambda_k) \end{bmatrix}^\top = \mathcal{N}(\mathbb{L}) \\ \mathbf{Bary}^{\text{num}} &= \begin{bmatrix} \gamma_1 \mathbf{n}(\lambda_1) & \cdots & \gamma_k \mathbf{n}(\lambda_k) \end{bmatrix}^\top \end{aligned} \quad \text{and} \quad \gamma_j = \left(\prod_{i=1, i\neq j}^{k} (\lambda_j - \lambda_i) \right)^{-1}$$

Reconstruction

$$\mathbf{H}(s) = \frac{\sum_{j=1}^{k} \mathbf{Bary}_j^{\text{num}} \mathbf{B}_{\text{lag}}(s)_j}{\sum_{j=1}^{k} \mathbf{Bary}^{\text{den}} \mathbf{B}_{\text{lag}}(s)_j} \quad \text{with} \quad \mathbf{B}_{\text{lag}}(s) = \begin{bmatrix} \frac{1}{s-\lambda_1} & \cdots & \frac{1}{s-\lambda_k} \end{bmatrix}^\top$$

A.C. Antoulas and B.D.O Anderson, *"On the scalar rational interpolation problem"*, IMA Journal of Mathematical Control and Information, vol. 3(2-3), 1986.

A. C. Antoulas, I. V. Gosea and C. P-V., *"Variable decoupling and the Kolmogorov superposition theorem for rational functions"*, https://arxiv.org/abs/2605.07246.

Loewner rational approximation

Lagrangian interpolation without computation (slice of the result)

Data generated from $\mathbf{G}(s) = \dfrac{1}{s+3}$ of complexity (1), with $k = 2$, $q = 2$

$$\left.\begin{array}{rcl} \lambda_j & = & \begin{bmatrix} 1 & 2 \end{bmatrix} \\ \mu_i & = & \begin{bmatrix} \frac{3}{2} & \frac{5}{2} \end{bmatrix} \end{array}\right\} \xrightarrow{\mathbf{H}} \left\{\begin{array}{rcl} \mathbf{w}_j & = & \begin{bmatrix} \frac{1}{4} & \frac{1}{5} \end{bmatrix} \\ \mathbf{v}_i & = & \begin{bmatrix} \frac{2}{9} & \frac{2}{11} \end{bmatrix} \end{array}\right.$$

$$\mathbb{L} = \begin{bmatrix} -\frac{1}{18} & -\frac{2}{45} \\ -\frac{1}{22} & -\frac{2}{55} \end{bmatrix} \text{ and } \mathbf{c} = \begin{bmatrix} -\frac{4}{5} \\ 1 \end{bmatrix}$$

$$\boldsymbol{\gamma} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \mathbf{d}_\lambda = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \text{ and } \textbf{Bary}^{\text{den}} = \begin{bmatrix} -4 \\ 5 \end{bmatrix}$$

Similarly, using numerator, we get

$$\textbf{Bary}^{\text{num}} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \text{ and } \mathbf{B}_{\text{lag}}(s) = \begin{bmatrix} \frac{1}{s-1} \\ \frac{1}{s-2} \end{bmatrix}$$

Content

Forewords

Loewner rational approximation

Loewner realization

Conclusion

SISO interpolation problem

Given the right and left data (λ_j and μ_i are distinct):

$$\begin{array}{ll} \{\lambda_j, \mathbf{w}_j\} & j = 1, \ldots, k \\ \{\mu_i, \mathbf{v}_i^\top\} & i = 1, \ldots, q \end{array}$$

we seek $\mathcal{S} : (E, A, B, C)$, whose transfer function is $\mathbf{H}(s) = C(sE - A)^{-1}B$ s.t.

$$\begin{array}{lcll} \mathbf{H}(\lambda_j) & = & \mathbf{w}_j & j = 1, \ldots, k \\ \mathbf{H}(\mu_i) & = & \mathbf{v}_i^\top & i = 1, \ldots, q \end{array}$$

A.J. Mayo and A.C. Antoulas, *"A framework for the solution of the generalized realization problem"*, Linear Algebra and its Applications, vol. 425(2-3), 2007.

I.V. Gosea, C. P-V. and A.C. Antoulas, *"Data-driven modeling and control of large-scale dynamical systems in the Loewner framework"*, Handbook in Numerical Analysis, vol. 23, January 2022.

MIMO tangential interpolation problem

Given the right and left data (λ_j and μ_i are distinct):

$$\begin{array}{ll} \{\lambda_j, \mathbf{r}_j, \mathbf{w}_j\} & j = 1, \ldots, k \\ \{\mu_i, \mathbf{l}_i^\top, \mathbf{v}_i^\top\} & i = 1, \ldots, q \end{array}$$

we seek $\mathcal{S} : (E, A, B, C)$, whose transfer function is $\mathbf{H}(s) = C(sE - A)^{-1}B$ s.t.

$$\begin{array}{lcll} \mathbf{H}(\lambda_j)\mathbf{r}_j & = & \mathbf{w}_j & j = 1, \ldots, k \\ \mathbf{l}_i^\top \mathbf{H}(\mu_i) & = & \mathbf{v}_i^\top & i = 1, \ldots, q \end{array}$$

A.J. Mayo and A.C. Antoulas, *"A framework for the solution of the generalized realization problem"*, Linear Algebra and its Applications, vol. 425(2-3), 2007.

I.V. Gosea, C. P-V. and A.C. Antoulas, *"Data-driven modeling and control of large-scale dynamical systems in the Loewner framework"*, Handbook in Numerical Analysis, vol. 23, January 2022.

The right data can be expressed as:

$$\begin{aligned} \mathbf{\Lambda} &= \mathbf{diag}\ [\lambda_1, \ldots, \lambda_k] && \in \mathbb{C}^{k\times k} \\ \mathbf{R} &= \begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \ldots & \mathbf{r}_k \end{bmatrix} && \in \mathbb{C}^{n_u\times k} \\ \mathbb{W} &= \begin{bmatrix} \mathbf{w}_1 & \mathbf{w}_2 & \ldots & \mathbf{w}_k \end{bmatrix} && \in \mathbb{C}^{n_y\times k} \end{aligned}$$

and the left data can be expressed as:

$$\begin{aligned} \mathbf{M} &= \mathbf{diag}\ [\mu_1, \ldots, \mu_q] && \in \mathbb{C}^{q\times q} \\ \mathbf{L}^\top &= \begin{bmatrix} \mathbf{l}_1 & \mathbf{l}_2 & \ldots & \mathbf{l}_q \end{bmatrix} && \in \mathbb{C}^{n_y\times q} \\ \mathbb{V}^\top &= \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_q \end{bmatrix} && \in \mathbb{C}^{n_u\times q} \end{aligned}$$

The Loewner matrix in this case is

$$\mathbb{L} = \begin{bmatrix} \frac{\mathbf{v}_1^\top \mathbf{r}_1 - \mathbf{l}_1^\top \mathbf{w}_1}{\mu_1 - \lambda_1} & \cdots & \frac{\mathbf{v}_1^\top \mathbf{r}_k - \mathbf{l}_1^\top \mathbf{w}_k}{\mu_1 - \lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_q^\top \mathbf{r}_1 - \mathbf{l}_q^\top \mathbf{w}_1}{\mu_q - \lambda_1} & \cdots & \frac{\mathbf{v}_q^\top \mathbf{r}_k - \mathbf{l}_q^\top \mathbf{w}_k}{\mu_q - \lambda_k} \end{bmatrix} \in \mathbb{C}^{q\times k}$$

With this notation $\mathbb{L}$ satisfy the Sylvester equation : $\mathbf{M}\mathbb{L} - \mathbb{L}\mathbf{\Lambda} = \mathbb{V}\mathbf{R} - \mathbf{L}\mathbb{W}$.

The right data can be expressed as:

$$\begin{aligned} \mathbf{\Lambda} &= \mathbf{diag}\,[\lambda_1, \ldots, \lambda_k] && \in \mathbb{C}^{k\times k} \\ \mathbf{R} &= \begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \ldots & \mathbf{r}_k \end{bmatrix} && \in \mathbb{C}^{n_u\times k} \\ \mathbb{W} &= \begin{bmatrix} \mathbf{w}_1 & \mathbf{w}_2 & \ldots & \mathbf{w}_k \end{bmatrix} && \in \mathbb{C}^{n_y\times k} \end{aligned}$$

and the left data can be expressed as:

$$\begin{aligned} \mathbf{M} &= \mathbf{diag}\,[\mu_1, \ldots, \mu_q] && \in \mathbb{C}^{q\times q} \\ \mathbf{L}^\top &= \begin{bmatrix} \mathbf{l}_1 & \mathbf{l}_2 & \ldots & \mathbf{l}_q \end{bmatrix} && \in \mathbb{C}^{n_y\times q} \\ \mathbb{V}^\top &= \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_q \end{bmatrix} && \in \mathbb{C}^{n_u\times q} \end{aligned}$$

The Loewner matrix in this case is

$$\mathbb{L} = \begin{bmatrix} \frac{\mathbf{v}_1{}^\top \mathbf{r}_1 - \mathbf{l}_1^\top \mathbf{w}_1}{\mu_1 - \lambda_1} & \cdots & \frac{\mathbf{v}_1^\top \mathbf{r}_k - \mathbf{l}_1^\top \mathbf{w}_k}{\mu_1 - \lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_q^\top \mathbf{r}_1 - \mathbf{l}_q^\top \mathbf{w}_1}{\mu_q - \lambda_1} & \cdots & \frac{\mathbf{v}_q^\top \mathbf{r}_k - \mathbf{l}_q^\top \mathbf{w}_k}{\mu_q - \lambda_k} \end{bmatrix} \in \mathbb{C}^{q\times k}$$

With this notation $\mathbb{L}$ satisfy the Sylvester equation : $\mathbf{M}\mathbb{L} - \mathbb{L}\mathbf{\Lambda} = \mathbb{V}\mathbf{R} - \mathbf{L}\mathbb{W}$.

Loewner realization

Loewner pencil

The Loewner matrix is:

$$\mathbb{L} = \begin{bmatrix} \frac{\mathbf{v}_1{}^\top \mathbf{r}_1 - \mathbf{l}_1^\top \mathbf{w}_1}{\mu_1 - \lambda_1} & \cdots & \frac{\mathbf{v}_1^\top \mathbf{r}_k - \mathbf{l}_1^\top \mathbf{w}_k}{\mu_1 - \lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_q^\top \mathbf{r}_1 - \mathbf{l}_q^\top \mathbf{w}_1}{\mu_q - \lambda_1} & \cdots & \frac{\mathbf{v}_q^\top \mathbf{r}_k - \mathbf{l}_q^\top \mathbf{w}_k}{\mu_q - \lambda_k} \end{bmatrix} \in \mathbb{C}^{q \times k}$$

$$\mathbf{M}\mathbb{L} - \mathbb{L}\mathbf{\Lambda} = \mathbb{V}\mathbf{R} - \mathbf{L}\mathbb{W}$$

The shifted Loewner matrix is:

$$\mathbb{S} = \begin{bmatrix} \frac{\mu_1 \mathbf{v}_1^\top \mathbf{r}_1 - \mathbf{l}_1^\top \mathbf{w}_1 \lambda_1}{\mu_1 - \lambda_1} & \cdots & \frac{\mu_1 \mathbf{v}_1^\top \mathbf{r}_k - \mathbf{l}_1^\top \mathbf{w}_k \lambda_k}{\mu_1 - \lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mu_q \mathbf{v}_q^\top \mathbf{r}_1 - \mathbf{l}_q^\top \mathbf{w}_1 \lambda_1}{\mu_q - \lambda_1} & \cdots & \frac{\mu_q \mathbf{v}_q^\top \mathbf{r}_k - \mathbf{l}_q^\top \mathbf{w}_k \lambda_k}{\mu_q - \lambda_k} \end{bmatrix} \in \mathbb{C}^{q \times k}$$

$$\mathbf{M}\mathbb{S} - \mathbb{S}\mathbf{\Lambda} = \mathbf{L}\mathbb{V}\mathbf{R} - \mathbf{L}\mathbb{W}\mathbf{\Lambda}$$

If data are sampled from $\mathbf{G}(s) = C(sE - A)^{-1}B$, and recall the generalized tangential observability and controllability matrices:

$$\mathcal{O}_q = \begin{bmatrix} \mathbf{l}_1^\top C(\mu_1 E - A)^{-1} \\ \vdots \\ \mathbf{l}_q^\top C(\mu_q E - A)^{-1} \end{bmatrix} \in \mathbb{C}^{q \times n}, \quad \mathcal{R}_k = \Big[(\lambda_1 E - A)^{-1} B \mathbf{r}_1, \ldots, (\lambda_k E - A)^{-1} B \mathbf{r}_k, \Big] \in \mathbb{C}^{n \times k}$$

Then,

$$\begin{aligned} [\mathbb{L}]_{ij} &= \frac{\mathbf{v}_i^\top \mathbf{r}_j - \mathbf{l}_i^\top \mathbf{w}_j}{\mu_i - \lambda_j} \\ &= -\mathbf{l}_i^\top C(\mu_j E - A)^{-1} E (\lambda_i E - A)^{-1} B \mathbf{r}_j \\ &= -[\mathcal{O}_q]_i E [\mathcal{R}_k]_j \end{aligned}$$

$$\begin{aligned} [\mathbb{S}]_{ij} &= \frac{\mu_i \mathbf{v}_i^\top \mathbf{r}_j - \mathbf{l}_i^\top \mathbf{w}_j \lambda_j}{\mu_i - \lambda_j} \\ &= -\mathbf{l}_i^\top C(\mu_j E - A)^{-1} A (\lambda_i E - A)^{-1} B \mathbf{r}_j \\ &= -[\mathcal{O}_q]_i A [\mathcal{R}_k]_j \end{aligned}$$

If data are sampled from $\mathbf{G}(s) = C(sE - A)^{-1}B$, and recall the generalized tangential observability and controllability matrices:

$$\mathcal{O}_q = \begin{bmatrix} \mathbf{l}_1^\top C(\mu_1 E - A)^{-1} \\ \vdots \\ \mathbf{l}_q^\top C(\mu_q E - A)^{-1} \end{bmatrix} \in \mathbb{C}^{q \times n}, \quad \mathcal{R}_k = \Big[(\lambda_1 E - A)^{-1} B \mathbf{r}_1, \ldots, (\lambda_k E - A)^{-1} B \mathbf{r}_k, \Big] \in \mathbb{C}^{n \times k}$$

Then,

$$\begin{aligned} [\mathbb{L}]_{ij} &= \frac{\mathbf{v}_i^\top \mathbf{r}_j - \mathbf{l}_i^\top \mathbf{w}_j}{\mu_i - \lambda_j} \\ &= -\mathbf{l}_i^\top C(\mu_j E - A)^{-1} E (\lambda_i E - A)^{-1} B \mathbf{r}_j \\ &= -[\mathcal{O}_q]_i E [\mathcal{R}_k]_j \end{aligned}$$

$$\begin{aligned} [\mathbb{S}]_{ij} &= \frac{\mu_i \mathbf{v}_i^\top \mathbf{r}_j - \mathbf{l}_i^\top \mathbf{w}_j \lambda_j}{\mu_i - \lambda_j} \\ &= -\mathbf{l}_i^\top C(\mu_j E - A)^{-1} A (\lambda_i E - A)^{-1} B \mathbf{r}_j \\ &= -[\mathcal{O}_q]_i A [\mathcal{R}_k]_j \end{aligned}$$

If data are sampled from $\mathbf{G}(s) = C(sE - A)^{-1}B$, and recall the generalized tangential observability and controllability matrices:

$$\mathcal{O}_q = \begin{bmatrix} \mathbf{l}_1^\top C(\mu_1 E - A)^{-1} \\ \vdots \\ \mathbf{l}_q^\top C(\mu_q E - A)^{-1} \end{bmatrix} \in \mathbb{C}^{q \times n}, \quad \mathcal{R}_k = \begin{bmatrix} (\lambda_1 E - A)^{-1} B \mathbf{r}_1, \ldots, (\lambda_k E - A)^{-1} B \mathbf{r}_k, \end{bmatrix} \in \mathbb{C}^{n \times k}$$

Then,

$$\begin{aligned} [\mathbb{L}]_{ij} &= \frac{\mathbf{v}_i^\top \mathbf{r}_j - \mathbf{l}_i^\top \mathbf{w}_j}{\mu_i - \lambda_j} \\ &= -\mathbf{l}_i^\top C(\mu_j E - A)^{-1} E (\lambda_i E - A)^{-1} B \mathbf{r}_j \\ &= -[\mathcal{O}_q]_i E [\mathcal{R}_k]_j \end{aligned}$$

$$\begin{aligned} [\mathbb{S}]_{ij} &= \frac{\mu_i \mathbf{v}_i^\top \mathbf{r}_j - \mathbf{l}_i^\top \mathbf{w}_j \lambda_j}{\mu_i - \lambda_j} \\ &= -\mathbf{l}_i^\top C(\mu_j E - A)^{-1} A (\lambda_i E - A)^{-1} B \mathbf{r}_j \\ &= -[\mathcal{O}_q]_i A [\mathcal{R}_k]_j \end{aligned}$$

Lemma

Assume that $k = q$, then $\mathbf{H}(s) = C(sE - A)^{-1}B$ with

$$E = -\mathbb{L}, \quad A = -\mathbb{S}, \quad , \; B = \mathbb{V}, \quad C = \mathbb{W},$$

is **a descriptor realization** interpolating the data.

Theorem: redundant data via SVD

A realization $(\hat{E}, \hat{A}, \hat{B}, \hat{C})$ of an (approximate) interpolant is given by:

$$\hat{E} = -Y^H \mathbb{L} X, \quad \hat{A} = -Y^H \mathbb{S} X, \hat{B} = -Y^H \mathbb{V}, \hat{C} = \mathbb{W} X$$

where

$$[\mathbb{L}, \; \mathbb{S}] = Y \Sigma_l \tilde{X}^H, \quad \begin{bmatrix} \mathbb{L} \\ \mathbb{S} \end{bmatrix} = \tilde{Y} \Sigma_r X^H, \; Y \in \mathbb{C}^{q \times r}, X \in \mathbb{C}^{k \times r}.$$

Lemma

Assume that $k = q$, then $\mathbf{H}(s) = C(sE - A)^{-1}B$ with

$$E = -\mathbb{L}, \quad A = -\mathbb{S}, \quad , \; B = \mathbb{V}, \quad C = \mathbb{W},$$

is **a descriptor realization** interpolating the data.

Theorem: redundant data via SVD

A realization $(\hat{E}, \hat{A}, \hat{B}, \hat{C})$ of an (approximate) interpolant is given by:

$$\hat{E} = -Y^H \mathbb{L} X, \quad \hat{A} = -Y^H \mathbb{S} X, \hat{B} = -Y^H \mathbb{V}, \hat{C} = \mathbb{W} X$$

where

$$[\mathbb{L}, \; \mathbb{S}] = Y \Sigma_l \tilde{X}^H, \quad \begin{bmatrix} \mathbb{L} \\ \mathbb{S} \end{bmatrix} = \tilde{Y} \Sigma_r X^H, \; Y \in \mathbb{C}^{q \times r}, X \in \mathbb{C}^{k \times r}.$$

Loewner realization

Connection with the projection (Petrov-Galerkin) framework

Being known the full state realization $\mathcal{S}$

$$\begin{aligned} E\dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + B\mathbf{u}(t) \\ \mathbf{y}(t) &= C\mathbf{x}(t) + D\mathbf{u}(t) \end{aligned}$$

The Petrov-Galerkin reduced order model $\hat{\mathcal{S}}$ is obtained with projectors $W, V \in \mathbb{C}^{n\times r}$ as

$$\begin{aligned} W^\top EV\dot{\hat{\mathbf{x}}}(t) &= W^\top AV\hat{\mathbf{x}}(t) + W^\top B\mathbf{u}(t) \\ \hat{\mathbf{y}}(t) &= CV\hat{\mathbf{x}}(t) + D\mathbf{u}(t) \end{aligned}$$

The above equations assume $W = \textbf{colspan}(\mathcal{W}), V = \textbf{colspan}(\mathcal{V})$ selected such that they meet the approximation objective:

$$\mathbf{x}(t) \approx V\hat{\mathbf{x}}(t) \quad \text{and} \quad \hat{\mathbf{x}}(0) = W^\top \mathbf{x}(0)$$

Remark

Choosing two different bases V' and W' that respectively span the same subspaces $\mathcal{V}$ and $\mathcal{W}$ result in the same reconstructed solution $\mathbf{x}(t)$. Thus, subspaces are relevant, not basis.

Loewner realization

Connection with the projection (Petrov-Galerkin) framework

Now recall,

$$-\mathcal{O}E\mathcal{R} = \mathbb{L} \quad \text{and} \quad -\mathcal{O}A\mathcal{R} = \mathbb{S},$$

where the generalized reachability and observability matrices are (complex conjugation trick for realness)

$$\mathcal{R} = \begin{bmatrix} \mathbf{r}_1^\top B^\top (\lambda_1 E - A)^{-\top} \\ \bar{\mathbf{r}}_1^T B^\top (\bar{\lambda}_1 E - A)^{-\top} \\ \vdots \\ \mathbf{r}_{k/2}^\top B^\top (\lambda_{k/2} E - A)^{-\top} \\ \bar{\mathbf{r}}_{k/2} B^\top (\bar{\lambda}_{k/2} E - A)^{-\top} \end{bmatrix}^\top \quad \text{and} \quad \mathcal{O} = \begin{bmatrix} \mathbf{l}_1 C (\mu_1 E - A)^{-1} \\ \bar{\mathbf{l}}_1 C (\bar{\mu}_1 E - A)^{-1} \\ \vdots \\ \mathbf{l}_{q/2} C (\mu_{q/2} E - A)^{-1} \\ \bar{\mathbf{l}}_{q/2} C (\bar{\mu}_{q/2} E - A)^{-1} \end{bmatrix}.$$

Or equivalently

$$-(Y^\top J^H \mathcal{O})^\top E(\mathcal{R}JX) = \mathbb{L} \quad \text{and} \quad -(Y^\top J^H \mathcal{O})^\top A(\mathcal{R}JX) = \mathbb{S}$$

$$J_0 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & \imath \\ 1 & -\imath \end{bmatrix} \quad \text{and} \quad J = I_{k/2} \otimes J_0,$$

Loewner realization

Connection with the projection (Petrov-Galerkin) framework

Now recall,

$$-\mathcal{O}E\mathcal{R} = \mathbb{L} \quad \text{and} \quad -\mathcal{O}A\mathcal{R} = \mathbb{S},$$

where the generalized reachability and observability matrices are (complex conjugation trick for realness)

$$\mathcal{R} = \begin{bmatrix} \mathbf{r}_1^\top B^\top (\lambda_1 E - A)^{-\top} \\ \bar{\mathbf{r}}_1^T B^\top (\bar{\lambda}_1 E - A)^{-\top} \\ \vdots \\ \mathbf{r}_{k/2}^\top B^\top (\lambda_{k/2} E - A)^{-\top} \\ \bar{\mathbf{r}}_{k/2} B^\top (\bar{\lambda}_{k/2} E - A)^{-\top} \end{bmatrix}^\top \quad \text{and} \quad \mathcal{O} = \begin{bmatrix} \mathbf{l}_1 C (\mu_1 E - A)^{-1} \\ \bar{\mathbf{l}}_1 C (\bar{\mu}_1 E - A)^{-1} \\ \vdots \\ \mathbf{l}_{q/2} C (\mu_{q/2} E - A)^{-1} \\ \bar{\mathbf{l}}_{q/2} C (\bar{\mu}_{q/2} E - A)^{-1} \end{bmatrix}.$$

Or equivalently

$$-(Y^\top J^H \mathcal{O})^\top E(\mathcal{R} J X) = \mathbb{L} \quad \text{and} \quad -(Y^\top J^H \mathcal{O})^\top A(\mathcal{R} J X) = \mathbb{S}$$

$$J_0 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & \imath \\ 1 & -\imath \end{bmatrix} \quad \text{and} \quad J = I_{k/2} \otimes J_0,$$

Loewner realization

Connection with the projection (Petrov-Galerkin) framework

Therefore,

$$\underbrace{(Y^\top J^H \mathcal{O})^\top}_{W^\top} E \underbrace{(\mathcal{R}JX)}_{V} = \mathbb{L}, \quad \text{and} \quad \underbrace{(Y^\top J^H \mathcal{O})^\top}_{W^\top} A \underbrace{(\mathcal{R}JX)}_{V} = \mathbb{S},$$

Therefore,

$$W = Y^\top J^H \mathcal{O}, \text{ and } W \leftarrow V(W^\top V)^{-\top} \quad \text{and} \quad V = \mathcal{R}JX.$$

and

$$\mathbf{x}(t) \approx V\hat{\mathbf{x}}(t) = \hat{\mathbf{x}}_\uparrow(t)$$

can be recovered (approximated) solely from ROM simulation.

J. Toledo-Zucco, D. Matignon, C. P-V. and Y. Le Gorrec, *"Structure-preserving discretization and model order reduction of boundary-controlled 1D port-Hamiltonian systems"*, Systems & Control Letters, Vol. 194, December 2024.

M. Gouzien, C. P-V., G. Haine and D. Matignon, *"APort-Hamiltonian reduced order modelling of the 2D Maxwell equations"*, journal for Computation and Mathematics in Electrical and Electronic Engineering, 2025.

Loewner realization

Summary

Given $\{\lambda_j, \mathbf{r}_j, \mathbf{w}_j\}$ and $\{\mu_i, \mathbf{l}_i, \mathbf{v}_i\}$, seek $\mathbf{H}$ s.t.

$$\mathbf{H}(\lambda_j)\mathbf{r}_j = \mathbf{w}_j \text{ and } \mathbf{l}_i\mathbf{H}(\mu_i) = \mathbf{v}_i$$

$j = 1, \ldots, k$ and $i = 1, \ldots, q$.

Rational interpolation

$$\begin{aligned} \mathbf{H}(s) &= \mathbb{W}(-s\mathbb{L} + \mathbb{S})^{-1}\mathbb{V} \\ \hat{\mathbf{H}}(s) &= \hat{C}(s\hat{E} - \hat{A})^{-1}\hat{B} \end{aligned}$$

- underlying rational (r) order

$$\begin{aligned} r &= \mathbf{rank}\ (\boldsymbol{\xi}\mathbb{L} - \mathbb{S}) \\ &= \mathbf{rank}\ ([\mathbb{L}, \mathbb{S}]) \\ &= \mathbf{rank}\ ([\mathbb{L}^H, \mathbb{S}^H]^H) \end{aligned}$$

- and McMillan (ν) order

$$\nu = \mathbf{rank}\ (\mathbb{L})$$

- $\mathbb{L}$ and $\mathbb{S}$ are input-output independents.
- Minimal realization
- Compression by Loewner pencil SVD
- If $\mathcal{S}$ is known, then internal state $\mathbf{x}(t) \in \mathbb{R}^n$ is recovered by

$$\mathbf{x}(t) \approx V\hat{\mathbf{x}}(t) = \hat{\mathbf{x}}_{\uparrow}(t) \quad \text{where} \quad V = \mathcal{R}JX.$$

A.C. Antoulas, S. Lefteriu and A.C. Ionita, *"Chapter 8: A Tutorial Introduction to the Loewner Framework for Model Reduction"*, Model Reduction and Approximation: Theory and Algorithms, 2016.

Loewner realization

Edifying example #1

Data generated from $\mathbf{G}(s) = \frac{1}{s^2 + \frac{s}{100} + 1}$ of complexity 2, $n_u = 1$, $n_y = 1$, with $k = 2$, $q = 2$

$$\left.\begin{array}{rcl} \lambda_j & = & \begin{bmatrix} 2.0 & 4.0 \end{bmatrix} \\ \mu_i & = & \begin{bmatrix} 1.0 & 3.0 \end{bmatrix} \end{array}\right\} \xrightarrow{\mathbf{H}} \left\{\begin{array}{rcl} \mathbf{w}_j & = & \begin{bmatrix} 0.2 & 0.059 \end{bmatrix} \\ \mathbf{v}_i & = & \begin{bmatrix} 0.5 & 0.1 \end{bmatrix} \end{array}\right.$$

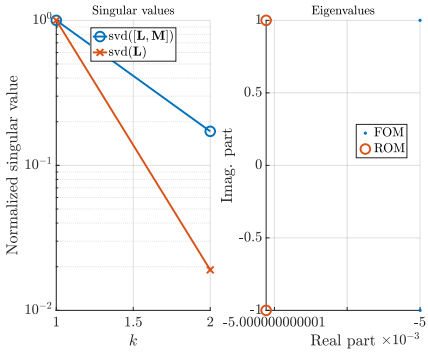
$$\mathbb{L} = \begin{bmatrix} -0.3 & -0.15 \\ -0.1 & -0.041 \end{bmatrix} \quad \mathbb{S} = \begin{bmatrix} -0.099 & -0.088 \\ -0.099 & -0.064 \end{bmatrix} \quad \mathbb{V}^\top = \begin{bmatrix} 0.5 & 0.1 \end{bmatrix} \quad \mathbb{W} = \begin{bmatrix} 0.2 & 0.059 \end{bmatrix}$$

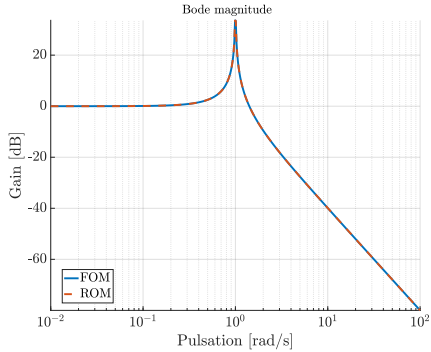
$$\hat{E} = \begin{bmatrix} 0.35 & -0.015 \\ 0.028 & 5.5 \cdot 10^{-3} \end{bmatrix} \quad \hat{A} = \begin{bmatrix} 0.16 & 0.032 \\ -0.06 & 2.5 \cdot 10^{-3} \end{bmatrix} \quad \hat{B} = \begin{bmatrix} 0.5 \\ 0.098 \end{bmatrix} \quad \hat{C} = \begin{bmatrix} 0.2 & -0.042 \end{bmatrix}$$

$$\hat{\mathbf{H}}(s) = \hat{C}(s\hat{E} - \hat{A})^{-1}\hat{B} \quad \text{and} \quad V = \begin{bmatrix} 0.46 & 0.019 \\ 0.2 & -0.042 \end{bmatrix}$$

Loewner realization

Edifying example #1

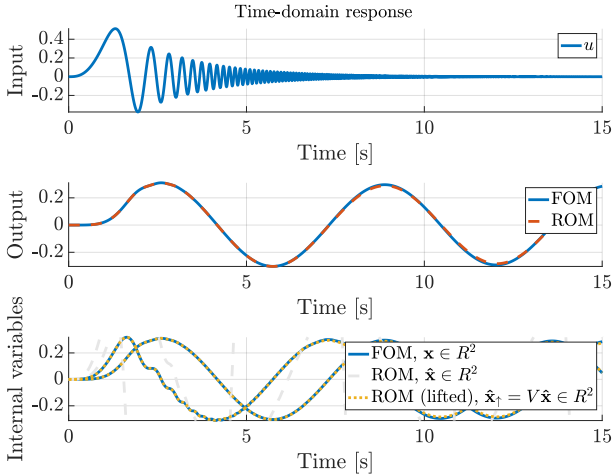




$$\hat{\mathbf{H}}(s) = \frac{1.0}{1.0\,s^2 + 0.01\,s + 1.0}$$

Loewner realization

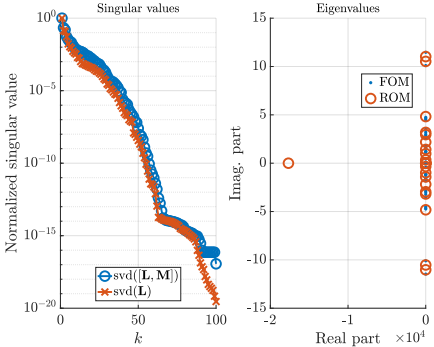
Edifying example #1

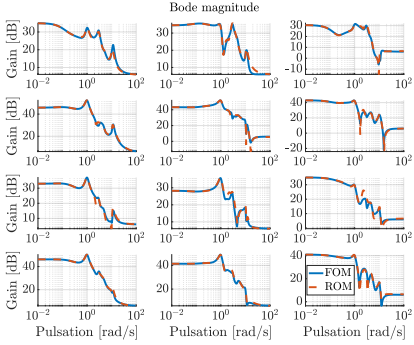


Loewner realization

Edifying example #2

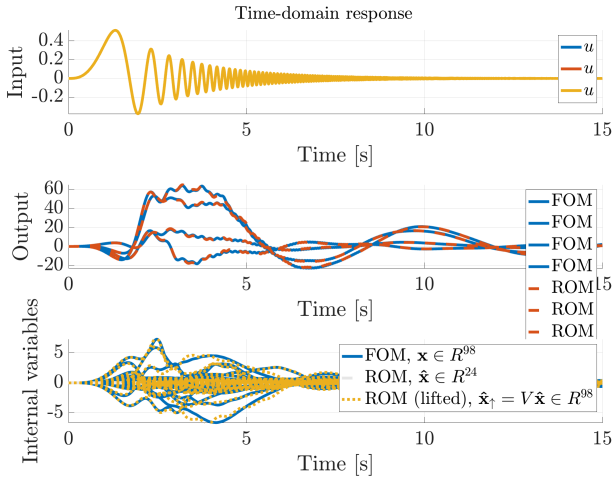
Data generated from $\mathbf{G}(s)$ of complexity 98, $n_u = 3$, $n_y = 4$, with $k = 100$, $q = 100$ and $r = 24$.





Loewner realization

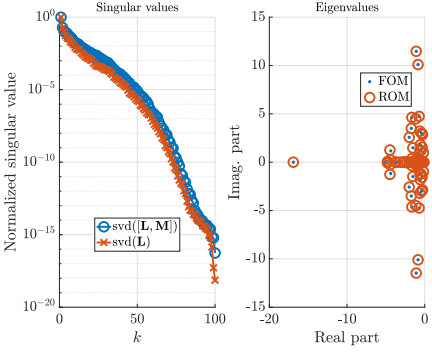
Edifying example #2

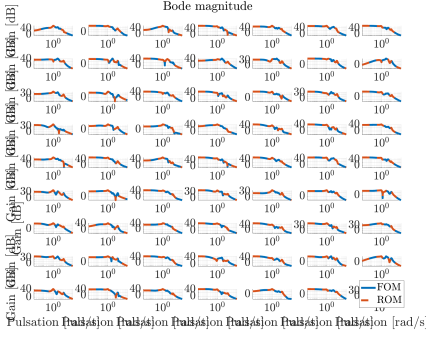


Loewner realization

Edifying example #3

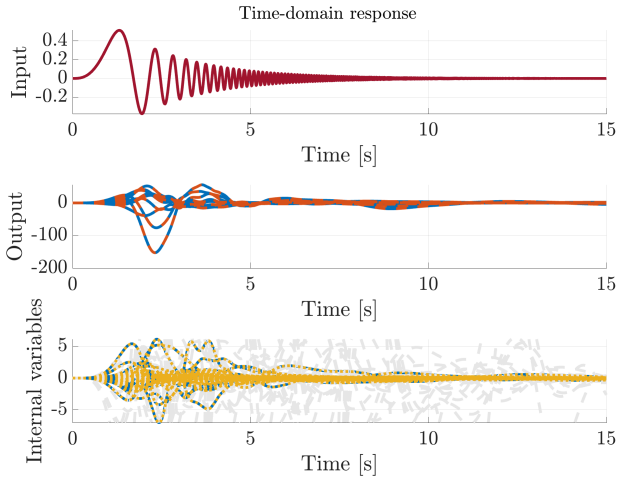
Data generated from $\mathbf{G}(s)$ of complexity (98), $n_u = 7$, $n_y = 9$, with $k = 100$, $q = 100$





Loewner realization

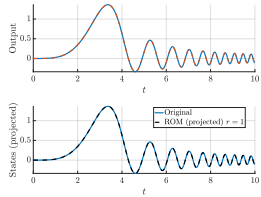
Edifying example #3

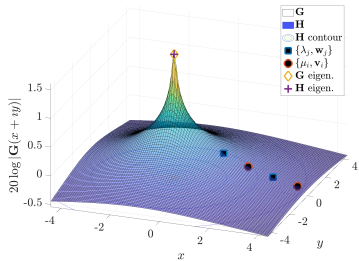


Loewner realization

Code break (https://github.com/cpoussot/lf)

```
% Model (very simple)
S           = ss(tf(2,[1 1]));
[A,B,C,D,E] = dssdata(S);
G           = @(s) C*((s*E-A)\B)+D;
% IP, responses, tangent
la  = [1 3];
mu  = [2 4];
k   = length(la);
q   = length(mu);
R   = ones(1,k);
L   = ones(q,1);
for ii = 1:k; W(1,1,ii) = G(la(ii)); end
for ii = 1:q; V(1,1,ii) = G(mu(ii)); end
% Loewner
[hr,mdl] = lf.loewner_tng(la,mu,W,V,R,L);
% Symbolic result
syms s
simplify(mdl.Cr*((mdl.Er*s-mdl.Ar)\mdl.Br)+mdl.Dr)
```





Passivity

The $m \times m$ transfer $\mathbf{H}(s)$ and spectral $\mathbf{\Phi_H}(s) = \mathbf{H}(s) + \mathbf{H}^\top(-s)$ functions with realizations

$$\left\{ \begin{array}{rcl} \dot{\mathbf{x}} & = & A\mathbf{x} + B\mathbf{u} \\ \mathbf{y} & = & C\mathbf{x} + D\mathbf{u} \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{rcl} \dot{\mathbf{x}}_{\mathbf{\Phi}} & = & \begin{bmatrix} A & 0 \\ 0 & -A^\top \end{bmatrix} \mathbf{x}_{\mathbf{\Phi}} + \begin{bmatrix} B \\ C^\top \end{bmatrix} \mathbf{u} \\ \mathbf{y}_{\mathbf{\Phi}} & = & \begin{bmatrix} C & -B^\top \end{bmatrix} \mathbf{x}_{\mathbf{\Phi}} + (D + D^\top)\mathbf{u} \end{array} \right.$$

satisfy $\mathbf{\Phi_H}(\imath\omega) > 0$, finite $\mathtt{Re}(\lambda_A) < 0$ and $D \succ 0$.

Spectral zeros

The zeros of $\mathbf{\Phi_H}$ (spectral zeros $\lambda_{\mathbf{\Phi}}$) are the solution of the GEVP of dimension $2n + m$

$$\begin{bmatrix} A & 0 & B \\ 0 & -A^\top & C^\top \\ C & -B^\top & D + D^\top \end{bmatrix} \begin{bmatrix} p_j \\ q_j \\ \mathbf{r}_j \end{bmatrix} = \lambda_j \begin{bmatrix} E & 0 & 0 \\ 0 & E^\top & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_j \\ q_j \\ \mathbf{r}_j \end{bmatrix}$$

Passivity

The $m \times m$ transfer $\mathbf{H}(s)$ and spectral $\mathbf{\Phi_H}(s) = \mathbf{H}(s) + \mathbf{H}^\top(-s)$ functions with realizations

$$\left\{ \begin{array}{rcl} \dot{\mathbf{x}} & = & A\mathbf{x} + B\mathbf{u} \\ \mathbf{y} & = & C\mathbf{x} + D\mathbf{u} \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{rcl} \dot{\mathbf{x}}_{\mathbf{\Phi}} & = & \begin{bmatrix} A & 0 \\ 0 & -A^\top \end{bmatrix} \mathbf{x}_{\mathbf{\Phi}} + \begin{bmatrix} B \\ C^\top \end{bmatrix} \mathbf{u} \\ \mathbf{y}_{\mathbf{\Phi}} & = & \begin{bmatrix} C & -B^\top \end{bmatrix} \mathbf{x}_{\mathbf{\Phi}} + (D + D^\top)\mathbf{u} \end{array} \right.$$

satisfy $\mathbf{\Phi_H}(\imath\omega) > 0$, finite $\mathtt{Re}(\lambda_A) < 0$ and $D \succ 0$.

pH

The $m \times m$ transfert passive $\mathbf{H}(s)$ transfer function admit a pH form

$$\left\{ \begin{array}{rcl} \dot{\mathbf{x}} & = & (J - R)Q\mathbf{x} + (G - P)\mathbf{u} \\ \mathbf{y} & = & (G + P)^\top Q\mathbf{x} + (N + S)\mathbf{u} \end{array} \right. \quad \text{with} \quad \mathcal{V} = \begin{bmatrix} -J & -G \\ G^\top & N \end{bmatrix} \quad \text{and} \quad \mathcal{W} = \begin{bmatrix} R & P \\ P^\top & S \end{bmatrix}$$

satisfy $\mathcal{V} = -\mathcal{V}^\top$, $\mathcal{W} = \mathcal{W}^\top \succeq 0$ and $Q = Q^\top \succeq 0$

Passivity to pH

$$\left.\begin{array}{rcl}\dot{\mathbf{x}} &=& A\mathbf{x}+B\mathbf{u}\\ \mathbf{y} &=& C\mathbf{x}+D\mathbf{u}\end{array}\right\} \quad\Longrightarrow\quad \left\{\begin{array}{rcl}\dot{\mathbf{x}} &=& (J-R)Q\mathbf{x}+(G-P)\mathbf{u}\\ \mathbf{y} &=& (G+P)^{\top}Q\mathbf{x}+(N+S)\mathbf{u}\end{array}\right.$$

Setting

$$\mathbf{S} := \begin{bmatrix} -A & -B \\ C & D \end{bmatrix},$$

one obtains the equivalent pH-form by solving

$$\frac{\mathbf{S}-\mathbf{S}^{\top}}{2} := \begin{bmatrix} -J & -G \\ G^{\top} & N \end{bmatrix} \quad \text{and} \quad \frac{\mathbf{S}+\mathbf{S}^{\top}}{2} := \begin{bmatrix} R & P \\ P^{\top} & S \end{bmatrix}.$$

P. Benner, P. Goyal and P. Van-Dooren, *"Identification of Port-Hamiltonian Systems from Frequency Response Data"*, Systems & Control Letters, vol. 143, 2020.

Transfer function

$$\mathbf{H}(s) = \frac{2\,s + 4}{s + 1}$$

ODE realization

$$\begin{aligned} \dot{\mathbf{x}} &= -1\mathbf{x} + 2\mathbf{u} \\ \mathbf{y} &= 1\mathbf{x} + 2\mathbf{u} \end{aligned}$$

Passivity

- $\mathtt{Re}(\Lambda(-1)) = -1 < 0$
- $D = 2 \succ 0$
- $\mathbf{\Phi_H}(\imath\omega) = \mathbf{H}(\imath\omega) + \mathbf{H}(-\imath\omega)^\top > 0$

$$\left.\begin{aligned} \mathbf{\Phi_H}(s) &= \tfrac{4\left(s^2-2\right)}{s^2-1} \\ \mathbf{\Phi_H}(\imath\omega) &= \tfrac{4\left(w^2+2\right)}{w^2+1} \end{aligned}\right\} \Rightarrow \lambda_{\mathbf{\Phi}} = \begin{bmatrix} -\sqrt{2} \\ \sqrt{2} \\ \infty \end{bmatrix}$$

Loewner realization

Passivity & port-Hamiltonian form (example)

Transfer function

$$\mathbf{H}(s) = \frac{2\,s+4}{s+1}$$

ODE realization

$$\begin{aligned} \dot{\mathbf{x}} &= -1\mathbf{x} + 2\mathbf{u} \\ \mathbf{y} &= 1\mathbf{x} + 2\mathbf{u} \end{aligned}$$

pH realization

$$\begin{aligned} \dot{\mathbf{x}} &= (J-R)Q\mathbf{x} + (G-P)\mathbf{u} \\ \mathbf{y} &= (G+P)^\top Q\mathbf{x} + (N+S)\mathbf{u} \end{aligned} \quad \text{or} \quad \begin{aligned} \dot{\mathbf{x}} &= (0-1)\,1\mathbf{x} + \left(\tfrac{3}{2} - \left(-\tfrac{1}{2}\right)\right)\mathbf{u} \\ \mathbf{y} &= \left(\tfrac{3}{2} + -\tfrac{1}{2}\right)\mathbf{x} + (0+2)\,\mathbf{u} \end{aligned}$$

where $\mathcal{V} = -\mathcal{V}^\top$, $\mathcal{W} = \mathcal{W}^\top \succeq 0$ and $Q = Q^\top \succeq 0$

$$\mathcal{V} = \begin{bmatrix} -J & -G \\ G^\top & N \end{bmatrix} = \begin{bmatrix} 0 & -\frac{3}{2} \\ \frac{3}{2} & 0 \end{bmatrix} \quad \text{and} \quad \mathcal{W} = \begin{bmatrix} R & P \\ P^\top & S \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 2 \end{bmatrix}$$

The right data can be expressed as:

$$\begin{aligned} \mathbf{\Lambda} &= \mathbf{diag}\,[\lambda_1, \ldots, \lambda_k] && \in \mathbb{C}^{k\times k} \\ \mathbf{R} &= \begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \ldots & \mathbf{r}_k \end{bmatrix} && \in \mathbb{C}^{m\times k} \\ \mathbb{W} &= \begin{bmatrix} \mathbf{w}_1 & \mathbf{w}_2 & \ldots & \mathbf{w}_k \end{bmatrix} && \in \mathbb{C}^{m\times k} \end{aligned}$$

and the left data can be expressed as:

$$\begin{aligned} \mathbf{M} &= -\Lambda^H && \in \mathbb{C}^{q\times q} \\ \mathbf{L}^\top &= \mathbf{R}^H && \in \mathbb{C}^{m\times q} \\ \mathbb{V}^\top &= -\mathbb{W}^H && \in \mathbb{C}^{m\times q} \end{aligned}$$



A.C. Antoulas, *"A new result on passivity preserving model reduction"*, Systems & Control Letters, vol. 54, 2005.

Theorem

Let interpolate the spectral zeros, $(\lambda_j, \mathbf{r}_j)$ from the standard Loewner (n zeros in the open right half-plane)

$$\begin{bmatrix} A & 0 & B \\ 0 & -A^\top & C^\top \\ C & -B^\top & D + D^\top \end{bmatrix} \begin{bmatrix} p_j \\ q_j \\ \mathbf{r}_j \end{bmatrix} = \lambda_j \begin{bmatrix} E & 0 & 0 \\ 0 & E^\top & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_j \\ q_j \\ \mathbf{r}_j \end{bmatrix}$$

then $\hat{\mathbf{H}}(s) \in \mathbb{C}^{m \times m}$ of McMillan degree k and $\hat{\mathcal{S}}$ is a normalized pH form and satisfies ($j = 1, \cdots, k$)

$$\begin{aligned} \mathbb{L} &\succ 0 \\ \hat{\mathbf{H}}(\infty) &= D \\ \hat{\mathbf{H}}(\lambda_j)\mathbf{r}_j &= \mathbf{w}_j \\ \mathbf{r}_j^H \hat{\mathbf{H}}(-\overline{\lambda_j}) &= -\mathbf{w}_j^H \\ \hat{D} + \hat{D}^\top &\succ 0 \end{aligned}$$

P. Benner, P. Goyal and P. Van-Dooren, *"Identification of Port-Hamiltonian Systems from Frequency Response Data"*, Systems & Control Letters, vol. 143, 2020.

Loewner realization

Loewner and pH form

Require $\{\lambda_j^0, \mathbf{r}_j^0, \mathbf{w}_j^0\}$, $\{\mu_i^0, \mathbf{l}_i^{0\top}, \mathbf{v}_i^{0\top}\}$, D, objective order r.

- Construct the r-th order Loewner interpolant $\hat{\mathbf{\Sigma}} := (-\mathbb{L}, -\mathbb{S}, \mathbb{V}, \mathbb{W}, 0_m)$
- Compute the equivalent formulation $\hat{\mathbf{\Sigma}} := (I, \hat{A}, \hat{B}, \hat{C}, \hat{D})$ with transfer $\hat{\mathbf{H}}$
- Compute spectral zeros and directions $(\lambda_j, \mathbf{r}_j)$ of $\hat{\mathbf{\Sigma}}$, and set $\mathbf{w}_j = \hat{\mathbf{H}}(\lambda_j)\mathbf{r}_j$, $\mathbf{r}_j^H \hat{\mathbf{H}}(-\overline{\lambda_j}) = -\mathbf{w}_j^H$
- Construct $\mathbb{L}$, $\mathbb{S}$, $\mathbb{V}$ and $\mathbb{W}$
- Compute Cholesky decomposition $\mathbb{L} = T^\top T$
- Construct $\hat{\mathbf{\Sigma}}_{\text{n-pH}} := (I_n, T\hat{A}T^{-1}, T\hat{B}, \hat{C}T^{-1}, \hat{D})$
- Construct $\hat{\mathbf{\Sigma}}_{\text{pH}} := (M, Q, J, R, G, P, N, S)$

Ensure $\hat{\mathbf{\Sigma}}_{\text{pH}}$ ensuring interpolatory conditions and strictly passive.

P. Benner, P. Goyal and P. Van-Dooren, *"Identification of Port-Hamiltonian Systems from Frequency Response Data"*, Systems & Control Letters, vol. 143, 2020.

C.P-V., D. Matignon, G. Haine and P. Vuillemin, *"Data-driven port-Hamiltonian structured identification for non-strictly passive systems"*, in Proceedings of European Control Conference (ECC), Bucharest, Romania, July 2023.

Deal with non strict passivity

This typically occurs when no direct feed-through term exist. As a consequence, the resulting spectral zeros exhibit zeros on the imaginary axis. The first and last bullets address this point.

Deal with stability

Loewner rational model $\hat{\mathbf{H}}$ may present unstable singularities. Therefore we suggest a *post stabilisation* onto $\mathcal{RH}_\infty$ as

$$P_\infty(\hat{\mathbf{H}}) = \arg \inf_{\mathbf{G} \in \mathcal{RH}_\infty} ||\hat{\mathbf{H}} - \mathbf{G}||_{\mathcal{L}_\infty}.$$

T. Breiten and B. Unger, *"Passivity preserving model reduction via spectral factorization"*, Automatica. vol. 142, pp. 110368, 2022.

M. Kohler, *"On the closest stable descriptor system in the respective spaces $\mathcal{RH}_2$ and $\mathcal{RH}_\infty$"*, Linear Algebra and its Applications, vol. 443, pp. 34-49, 2014.

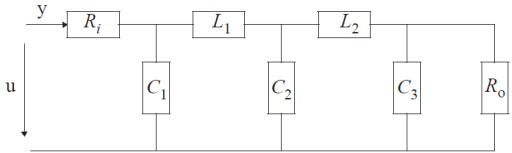
C.P-V., D. Matignon, G. Haine and P. Vuillemin, *"Data-driven port-Hamiltonian structured identification for non-strictly passive systems"*, in Proceedings of European Control Conference (ECC), Bucharest, Romania, July 2023.

Loewner realization

RLC ladder network (by S. Gugercin & A.C. Antoulas)

Variables & parameters

- x_1: voltage across C_1
- x_2: current across L_1
- x_3: voltage across C_2
- x_4: current across L_2
- x_5: voltage across C_3
- u: voltage
- y: current
- $C_i = \frac{1}{10}$ F
- $L_i = \frac{1}{10}$ H
- $R_i = \frac{1}{2}\Omega$, $R_o = 5\Omega$



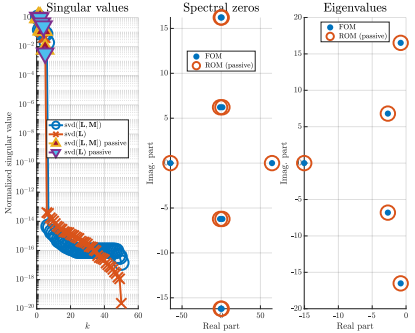
$$A = \begin{bmatrix} -20 & -10 & 0 & 0 & 0 \\ 10 & 0 & -10 & 0 & 0 \\ 0 & 10 & 0 & -10 & 0 \\ 0 & 0 & 10 & 0 & -10 \\ 0 & 0 & 0 & 10 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 20 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

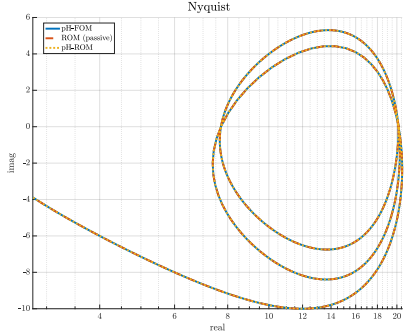
$$C = \begin{bmatrix} 20 & 0 & 0 & 0 & 0 \end{bmatrix} \quad D = 2$$

$$\mathbf{G}(s) = \frac{2.0\,s^5 + 440.0\,s^4 + 1.7 \cdot 10^3\,s^3 + 1.3 \cdot 10^5\,s^2 + 2.4 \cdot 10^5\,s + 4.4 \cdot 10^6}{s^5 + 22.0\,s^4 + 440.0\,s^3 + 6.6 \cdot 10^3\,s^2 + 3.8 \cdot 10^4\,s + 2.2 \cdot 10^5}$$

Loewner realization

RLC ladder network (by S. Gugercin & A.C. Antoulas)





Loewner realization

RLC ladder network (by S. Gugercin & A.C. Antoulas)

$$J = \begin{bmatrix} 0 & 5.7 & -2.4 & -9.6 & 2.4 \\ -5.7 & 0 & -16.0 & -1.5 & 0.38 \\ 2.4 & 16.0 & 0 & 0.94 & -0.24 \\ 9.6 & 1.5 & -0.94 & 0 & -6.3 \\ -2.4 & -0.38 & 0.24 & 6.3 & 0 \end{bmatrix} \quad G = \begin{bmatrix} 21.0 \\ 1.9 \\ -0.78 \\ -3.1 \\ 0.8 \end{bmatrix} \quad P = \begin{bmatrix} -6.4 \\ -0.99 \\ 0.63 \\ 1.9 \\ -0.61 \end{bmatrix}$$

$$R = \begin{bmatrix} 20.0 & 3.0 & -1.9 & -5.7 & 1.8 \\ 3.0 & 0.47 & -0.3 & -0.88 & 0.29 \\ -1.9 & -0.3 & 0.19 & 0.55 & -0.18 \\ -5.7 & -0.88 & 0.55 & 1.6 & -0.54 \\ 1.8 & 0.29 & -0.18 & -0.54 & 0.17 \end{bmatrix} \quad N = 0.0 \quad S = 2.0 \quad Q = I_5$$

Loewner realization

RLC ladder network (by S. Gugercin & A.C. Antoulas)

- $\mathcal{W} = \mathcal{W}^\top \succeq 0$

$$\mathcal{W} = \begin{bmatrix} 20.0 & 3.0 & -1.9 & -5.7 & 1.8 & -6.4 \\ 3.0 & 0.47 & -0.3 & -0.88 & 0.29 & -0.99 \\ -1.9 & -0.3 & 0.19 & 0.55 & -0.18 & 0.63 \\ -5.7 & -0.88 & 0.55 & 1.6 & -0.54 & 1.9 \\ 1.8 & 0.29 & -0.18 & -0.54 & 0.17 & -0.61 \\ -6.4 & -0.99 & 0.63 & 1.9 & -0.61 & 2.0 \end{bmatrix}$$

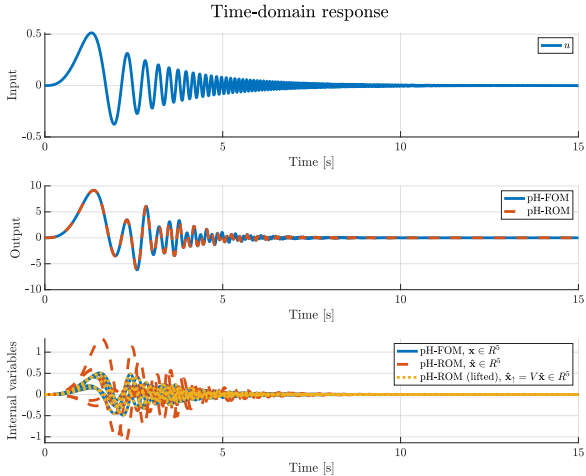
- $\mathcal{V} = -\mathcal{V}^\top$,

$$\mathcal{V} = \begin{bmatrix} 0 & -5.7 & 2.4 & 9.6 & -2.4 & -21.0 \\ 5.7 & 0 & 16.0 & 1.5 & -0.38 & -1.9 \\ -2.4 & -16.0 & 0 & -0.94 & 0.24 & 0.78 \\ -9.6 & -1.5 & 0.94 & 0 & 6.3 & 3.1 \\ 2.4 & 0.38 & -0.24 & -6.3 & 0 & -0.8 \\ 21.0 & 1.9 & -0.78 & -3.1 & 0.8 & 0 \end{bmatrix}$$

- $Q = Q^\top \succeq 0$

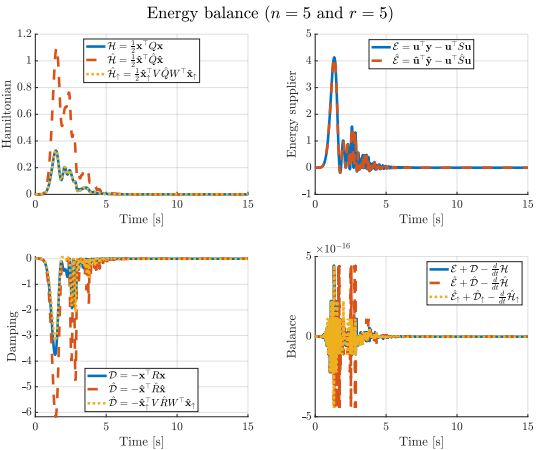
Loewner realization

RLC ladder network (by S. Gugercin & A.C. Antoulas)



Loewner realization

RLC ladder network (by S. Gugercin & A.C. Antoulas)



Time-domain lifting

Simulate and store $\hat{\mathbf{x}}(t)$, then lift thanks to generalized reachable subspace

$$\mathbf{x}(t) \approx V\hat{\mathbf{x}}_\uparrow(t)$$

As the system is a normalized passive one, the Hamiltonian energy $\mathcal{H}(t)$, internal dissipated power $\mathcal{D}(t)$ exchange power $\mathcal{E}(t)$ are given by

$$\begin{aligned}\mathcal{H}(t) &= \hat{\mathbf{x}}_\uparrow(t)^\top Q\hat{\mathbf{x}}_\uparrow(t)\\ \mathcal{D}(t) &= -\hat{\mathbf{x}}_\uparrow(t)^\top R\hat{\mathbf{x}}_\uparrow(t)\\ \mathcal{E}(t) &= \mathbf{u}(t)^\top\hat{\mathbf{y}}(t)\end{aligned}$$

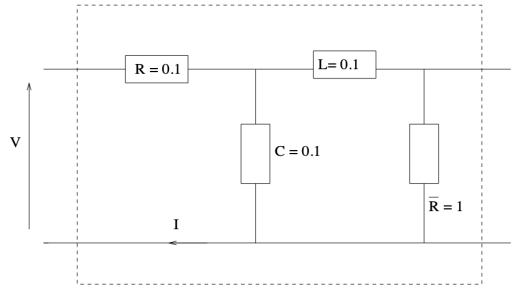
Loewner realization

RLC ladder network (by S. Gugercin)

Variables & parameters

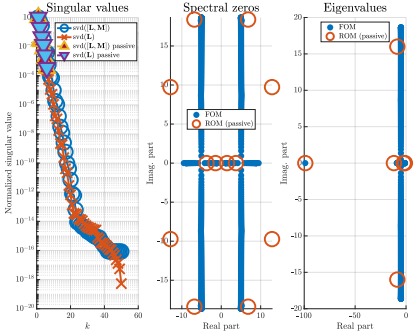
Circuit of multiple sections interconnected in cascade; each section is as shown in Figure. The input is the voltage V and the output is the current I, of the first section.

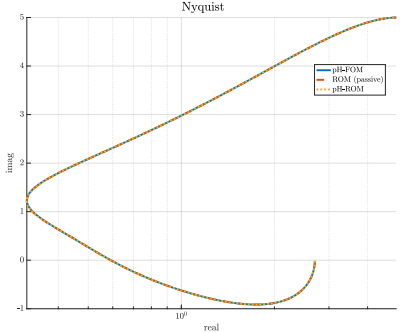
- x_i: voltage along C_i and current along L_i
- u: voltage V
- y: current I
- $n = 0.0$ internal variables
- $C_i = \frac{1}{10}$ F
- $L_i = \frac{1}{10}$ H
- $R_i = \frac{1}{10}\Omega$
- $\overline{R} = 1\Omega$



Loewner realization

RLC ladder network (by S. Gugercin)





Loewner realization

RLC ladder network (by S. Gugercin)

$$J = \begin{bmatrix} 0 & -29.0 & -29.0 & 29.0 & 19.0 & -10.0 \\ 29.0 & 0 & 12.0 & -12.0 & -7.8 & 4.4 \\ 29.0 & -12.0 & 0 & -17.0 & -9.5 & 5.3 \\ -29.0 & 12.0 & 17.0 & 0 & 8.5 & -4.7 \\ -19.0 & 7.8 & 9.5 & -8.5 & 0 & -2.0 \\ 10.0 & -4.4 & -5.3 & 4.7 & 2.0 & 0 \end{bmatrix} \quad G = \begin{bmatrix} -2.4 \\ 7.0 \\ 10.0 \\ -10.0 \\ -6.7 \\ 3.7 \end{bmatrix} \quad P = \begin{bmatrix} 28.0 \\ -12.0 \\ -14.0 \\ 13.0 \\ 5.4 \\ -3.9 \end{bmatrix}$$

$$R = \begin{bmatrix} 77.0 & -33.0 & -39.0 & 35.0 & 15.0 & -11.0 \\ -33.0 & 14.0 & 17.0 & -15.0 & -6.3 & 4.5 \\ -39.0 & 17.0 & 20.0 & -18.0 & -7.6 & 5.5 \\ 35.0 & -15.0 & -18.0 & 16.0 & 6.8 & -4.9 \\ 15.0 & -6.3 & -7.6 & 6.8 & 2.9 & -2.1 \\ -11.0 & 4.5 & 5.5 & -4.9 & -2.1 & 1.5 \end{bmatrix} \quad N = 0.0 \quad S = 10.0 \quad Q = I_6$$

Loewner realization

RLC ladder network (by S. Gugercin)

- $\mathcal{W} = \mathcal{W}^\top \succeq 0$

$$\mathcal{W} = \begin{bmatrix} 77.0 & -33.0 & -39.0 & 35.0 & 15.0 & -11.0 & 28.0 \\ -33.0 & 14.0 & 17.0 & -15.0 & -6.3 & 4.5 & -12.0 \\ -39.0 & 17.0 & 20.0 & -18.0 & -7.6 & 5.5 & -14.0 \\ 35.0 & -15.0 & -18.0 & 16.0 & 6.8 & -4.9 & 13.0 \\ 15.0 & -6.3 & -7.6 & 6.8 & 2.9 & -2.1 & 5.4 \\ -11.0 & 4.5 & 5.5 & -4.9 & -2.1 & 1.5 & -3.9 \\ 28.0 & -12.0 & -14.0 & 13.0 & 5.4 & -3.9 & 10.0 \end{bmatrix}$$

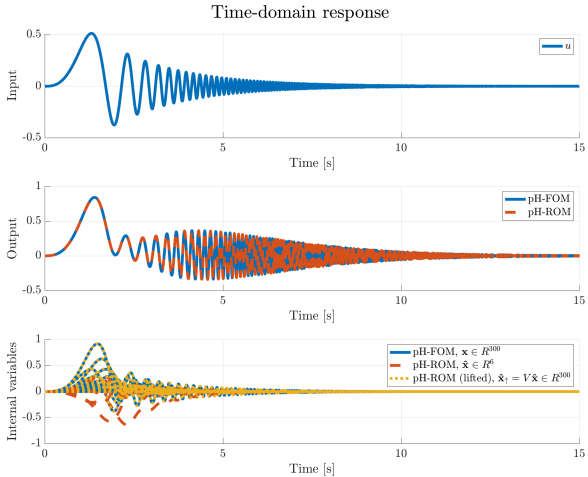
- $\mathcal{V} = -\mathcal{V}^\top$,

$$\mathcal{V} = \begin{bmatrix} 0 & 29.0 & 29.0 & -29.0 & -19.0 & 10.0 & 2.4 \\ -29.0 & 0 & -12.0 & 12.0 & 7.8 & -4.4 & -7.0 \\ -29.0 & 12.0 & 0 & 17.0 & 9.5 & -5.3 & -10.0 \\ 29.0 & -12.0 & -17.0 & 0 & -8.5 & 4.7 & 10.0 \\ 19.0 & -7.8 & -9.5 & 8.5 & 0 & 2.0 & 6.7 \\ -10.0 & 4.4 & 5.3 & -4.7 & -2.0 & 0 & -3.7 \\ -2.4 & 7.0 & 10.0 & -10.0 & -6.7 & 3.7 & 0 \end{bmatrix}$$

- $Q = Q^\top \succeq 0$

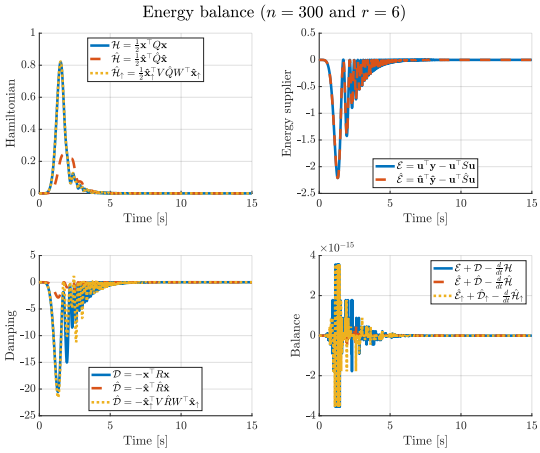
Loewner realization

RLC ladder network (by S. Gugercin)



Loewner realization

RLC ladder network (by S. Gugercin)



Time-domain lifting

Simulate and store $\hat{\mathbf{x}}(t)$, then lift thanks to generalized reachable subspace

$$\mathbf{x}(t) \approx V\hat{\mathbf{x}}_\uparrow(t)$$

As the system is a normalized passive one, the Hamiltonian energy $\mathcal{H}(t)$, internal dissipated power $\mathcal{D}(t)$ exchange power $\mathcal{E}(t)$ are given by

$$\begin{aligned}\mathcal{H}(t) &= \hat{\mathbf{x}}_\uparrow(t)^\top Q\hat{\mathbf{x}}_\uparrow(t)\\ \mathcal{D}(t) &= -\hat{\mathbf{x}}_\uparrow(t)^\top R\hat{\mathbf{x}}_\uparrow(t)\\ \mathcal{E}(t) &= \mathbf{u}(t)^\top\hat{\mathbf{y}}(t)\end{aligned}$$

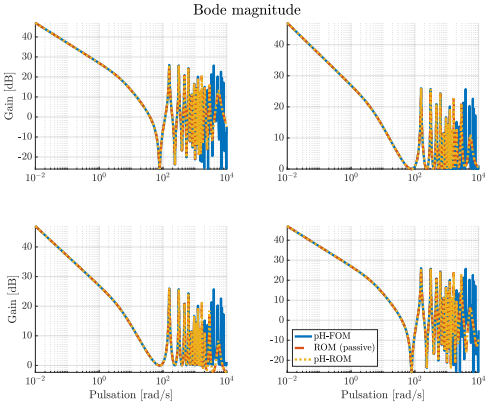
Loewner realization

Damped string (by D. Matignon & G. Haine)

Variables & parameters

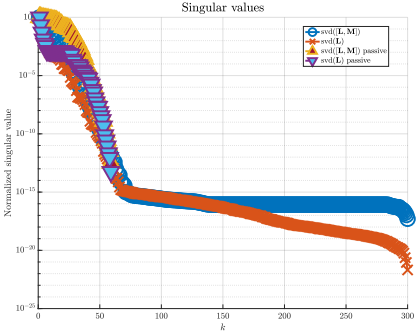
$$\mathbf{G}(s) = \frac{Z_c}{\sinh(d(s))} \begin{bmatrix} \cosh(d(s)) & -1 \\ -1 & \cosh(d(s)) \end{bmatrix}$$

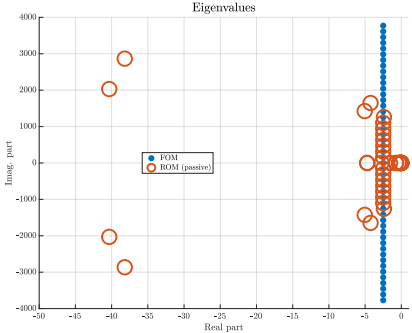
- $d(s) = \sqrt{(s\xi)^2 + (s\xi)\epsilon}$
- $\epsilon = \frac{1}{10}$
- $\xi = \frac{1}{50}$
- $Z_c = 1$
- $n = \infty$



Loewner realization

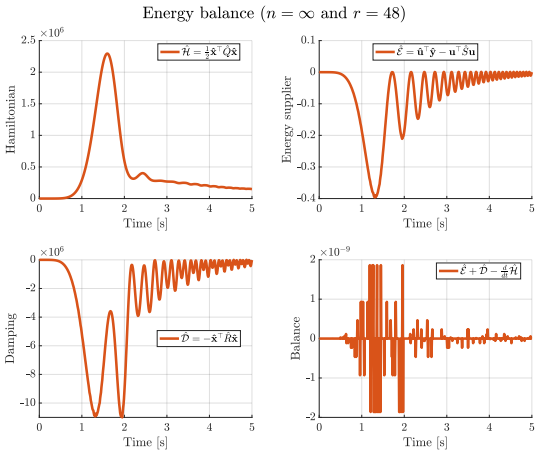
Damped string (by D. Matignon & G. Haine)





Loewner realization

Damped string (by D. Matignon & G. Haine)



Time-domain lifting

Simulate and store $\hat{\mathbf{x}}(t)$, then lift thanks to generalized reachable subspace

$$\mathbf{x}(t) \approx V\hat{\mathbf{x}}_{\uparrow}(t)$$

As the system is a normalized passive one, the Hamiltonian energy $\mathcal{H}(t)$, internal dissipated power $\mathcal{D}(t)$ exchange power $\mathcal{E}(t)$ are given by

$$\begin{array}{rcl} \mathcal{H}(t) & = & \hat{\mathbf{x}}_{\uparrow}(t)^{\top} Q \hat{\mathbf{x}}_{\uparrow}(t) \\ \mathcal{D}(t) & = & -\hat{\mathbf{x}}_{\uparrow}(t)^{\top} R \hat{\mathbf{x}}_{\uparrow}(t) \\ \mathcal{E}(t) & = & \mathbf{u}(t)^{\top} \hat{\mathbf{y}}(t) \end{array}$$

Loewner realization

Waveguide data-driven modeling (by M. Gouzien)

Content

Forewords

Loewner rational approximation

Loewner realization

Conclusion

Conclusion

Loewner... a versatile tool

- by interpolation...
- solves the LTI realization problem
- solves model/data-driven model reduction
- solves model/data-driven model approximation
- ... and pH
- ... and also, parametric, bilinear, quadratic...

  $\rightarrow$ direct impact in engineers and researchers life

... still so much to do

- Technical references, slides and code at
  https://cpoussot.github.io/
  https://github.com/cpoussot/LF
- MOR Digital Systems numerical suite
  http://mordigitalsystems.fr/

Conclusion

Loewner... a versatile tool

- by interpolation...
- solves the LTI realization problem
- solves model/data-driven model reduction
- solves model/data-driven model approximation
- ... and pH
- ... and also, parametric, bilinear, quadratic...

  $\rightarrow$ direct impact in engineers and researchers life

- **Technical references, slides and code at**
  `https://cpoussot.github.io/`
  `https://github.com/cpoussot/LF`
- **MOR Digital Systems numerical suite**
  `http://mordigitalsystems.fr/`

... still so much to do

The Loewner framework

A toolkit for identification, approximation, realization, projection...

C. Poussot-Vassal
November 28, 2026

Conclusion

Loewner... a versatile tool (self promotion)

Loewner & pH

- M. Gouzien, C. Poussot-Vassal, G. Haine and D. Matignon, "Port-Hamiltonian reduced order modelling of the 2D Maxwell equations", in journal for Computation and Mathematics in Electrical and Electronic Engineering (2020).
- J. Toledo-Zucco, D. Matignon, C. Poussot-Vassal and Y. Le Gorrec, "Structure-preserving discretization and model order reduction of boundary-controlled 1D port-Hamiltonian systems", in Systems & Control Letters, Vol. 194, December 2024, 105947.

Loewner & Loewner parametric

- A. C. Antoulas, I. V. Gosea and C. Poussot-Vassal, "On the Loewner framework, the Kolmogorov superposition theorem, and the curse of dimensionality", to appear in SIAM Review (Research Spotlight).
- T. Vojkovic, D. Quero, C. Poussot-Vassal and P. Vuillemin, "Low-order parametric state-space modeling of MIMO systems in the Loewner framework", in SIAM Journal on Applied Dynamical Systems, Vol. 22(4), November 2023, pp. 3130-3164.
- I. V. Gosea, C. Poussot-Vassal and A. C. Antoulas, "Data-driven modeling and control of large-scale dynamical systems in the Loewner framework", Handbook in Numerical Analysis, Vol. 23, January 2022, pp. 499-530.