MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

CSC

COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

The Loewner framework for parametric systems and the curse of dimensionality
Part II: Applications

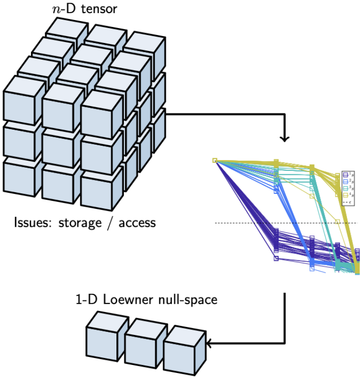
Ion Victor Gosea
joint work with A. C. Antoulas, Rice University, Houston
and with C. Poussot-Vassal, Onera, Toulouse

Model Reduction and Surrogate Modeling (MORe2024)
9-13 September 2024, La Jolla, California, USA

September 10th, 2024

Overview of the first talk

We presented **three main innovations** of the parametic Loewner framework.



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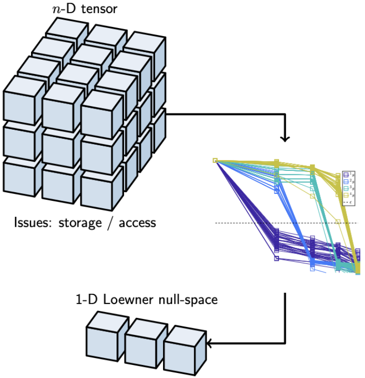

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Realizations can be constructed for any number of parameters n.

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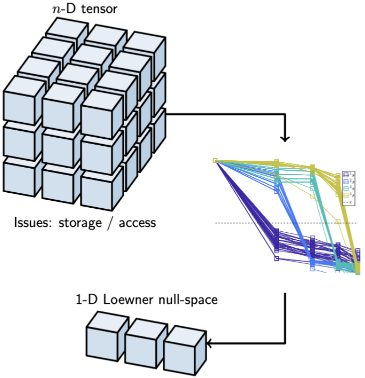

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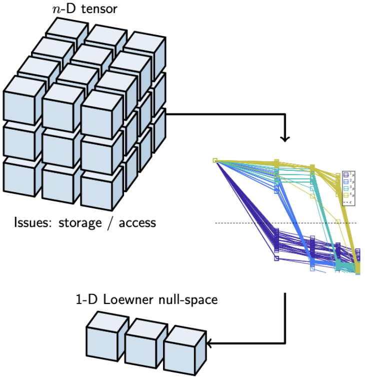

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The n-D Loewner matrix $\mathbb{L}_n$ is the solution of coupled Sylvester equations.

$\rightsquigarrow$ **The third innovation**

Compute the null space of $\mathbb{L}_n$ without explicitly computing the matrix $\mathbb{L}_n$.

Numerical Examples ⇝ Database of various test cases [1]

- These include both rational and irrational functions.
- For the rational ones, we go up to $n = 20$ parameters (with a low maximal power).

[1] A. P. Austin et al., Computer Physics Communications, "Practical algorithms for multivariate rational approximation", `https://doi.org/10.1016/j.cpc.2020.107663`

#4 Rational function

$$s_4{}^3 + \frac{s_1\, s_3}{s_3{}^2 + s_1 + s_2 + 1}$$

#5 Rational function

$$\frac{s_3{}^2 + s_1\, s_3\, s_5{}^3}{s_1{}^3 + s_4 + s_2\, s_3}$$

#6 Rational function

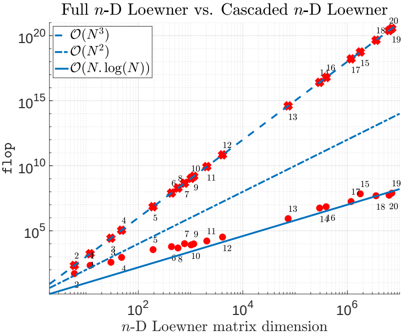
$$\frac{-\sqrt{2}\, s_6{}^2 + s_1 + s_3}{s_1{}^2 + s_4{}^3 + s_5{}^2 + s_6 + s_2\, s_3}$$

#7 Rational function

$$\frac{s_3\, s_2{}^3 + 1}{s_3\, s_2{}^2 + s_4{}^2 + s_6{}^3 + s_1 + s_5 + s_7}$$

#19 Rational function

$$\frac{3\, s_1{}^3 + s_{18}{}^2 + 4\, s_8 + s_{12} + s_{15} + s_{13}\, s_{14}}{s_3\, s_2{}^2 + \pi\, s_{16}{}^3 + s_{17}{}^2 + s_1 + s_4 + s_5 + s_6 + s_{13} + s_{19} + s_7\, s_8 + s_9\, s_{10}\, s_{11}}$$



#16 Arc-tangent function

$$\frac{\mathrm{atan}\left(x_1\right) + \mathrm{atan}\left(x_2\right) + \mathrm{atan}\left(x_3\right) + \mathrm{atan}\left(x_4\right)}{{x_1}^2\, {x_2}^2 - {x_1}^2 - {x_2}^2 + 1}$$

#17 Exponential function

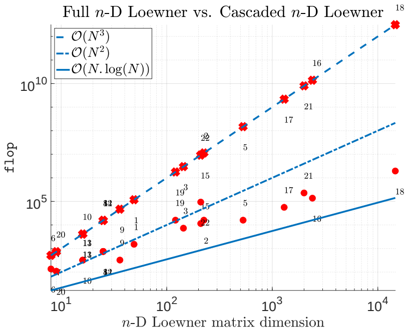
$$\frac{\mathrm{e}^{x_1\, x_2\, x_3\, x_4}}{{x_1}^2 + {x_2}^2 - x_3\, x_4 + 3}$$

#18 Sinc function

$$\frac{10\, \sin\left(x_1\right)\, \sin\left(x_2\right)\, \sin\left(x_3\right)\, \sin\left(x_4\right)}{x_1\, x_2\, x_3\, x_4}$$

#19 Sinc function

$$\frac{10\, \sin\left(x_1\right)\, \sin\left(x_2\right)}{x_1\, x_2}$$



https://sites.google.com/site/charlespoussotvassal/nd_loew_tcod.

Numerical Examples ⇝ Aircraft Flutter[2]

- Flutter refers to a dynamic **instability** of an elastic structure in a fluid flow.
- Example of flutter: the collapse of the Tacoma-Washington bridge in 1940, as a result of aeroelastic flutter caused by a 42 mph (68 km/h) wind.

[2] Models available from: A. dos Reis de Souza, C. Poussot-Vassal, P. Vuillemin, and J. Toledo Zucco, *Aircraft flutter suppression: from a parametric model to robust control*, ECC 2023, `https://ieeexplore.ieee.org/document/10178141`

Flutter objective: find a feedback control law which **extends** the **flutter-free** domain of the parameter values.

Typical parameters: Altitude, Mach number, Mass(es), Airspeed, etc.

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- In the 2D case, we assume that the Mach number and the mass are constant, thus the only parameter is the altitude.
- The aerodynamic and actuator parts are measured at few values using dedicated software:

$$\mathbf{Q}(i\omega_n, \theta_m), \ \mathbf{B}(i\omega_n, \theta_m), \quad n = 1, \cdots, N, \quad m = 1, \cdots, M.$$

- The matrices of the structural part are known for fixed parameter values: $\mathbf{M}(\theta_m)$, $\mathbf{D}(\theta_m)$, $\mathbf{K}(\theta_m)$; hence, the data for the problem are:

$$(s_n, \, \theta_m, \, \mathbf{\Phi}_{n,m}), \quad i = 1, \cdots, N, \quad j = 1, \cdots, M, \quad \text{where :}$$

$$\mathbf{\Phi}_{n,m} = \mathbf{C} \left(s_n^2 \mathbf{M}(\theta_m) + s_n \mathbf{D}(\theta_m) + \mathbf{K}(\theta_m) \right)^{-1} (\mathbf{Q}(s_n, \theta_m) + \mathbf{B}(s_n, \theta_m)), \quad s_n = i\omega_n.$$

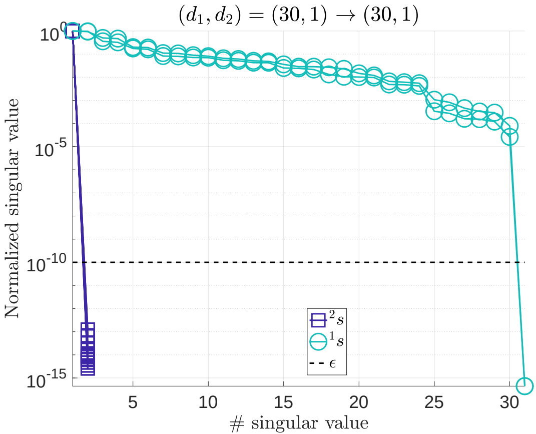
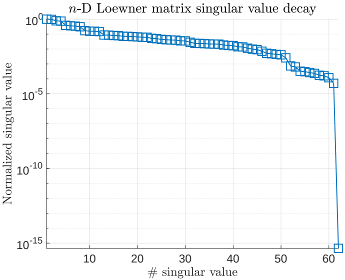
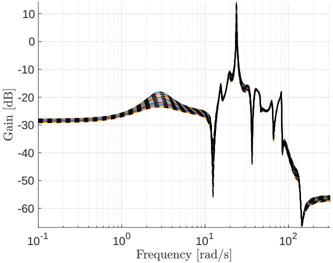
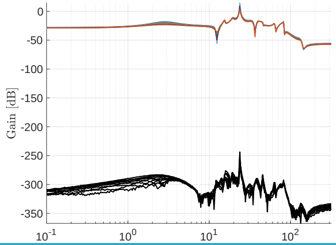


Figure: 2D flutter model: Loewner matrix singular values for frozen parameter (cyan) and frozen frequency (blue) ⇝ standard way to assign complexity of each parameter.


n-D Loewner matrix singular value decay
Normalized singular value
10^{0}
10^{-5}
10^{-10}
10^{-15}
10 20 30 40 50 60
singular value


Gain [dB]
10
0
-10
-20
-30
-40
-50
-60
10^{-1} 10^{0} 10^{1} 10^{2}
Frequency [rad/s]


Gain [dB]
0
-50
-100
-150
-200
-250
-300
-350
10^{-1} 10^{0} 10^{1} 10^{2}

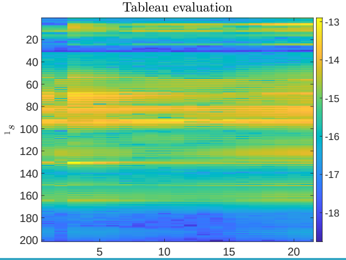


Tableau evaluation
l_s
20 40 60 80 100 120 140 160 180 200
5 10 15 20
-13 -14 -15 -16 -17 -18

Numerical Examples $\rightsquigarrow$ Aircraft Flutter extended in 3D & 7D[3]

[3] Acknowledgments to Pierre Vuillemin for (modified) data generation.

- This industrial example has a mixed model/data configuration.
- It models the **flutter phenomena** for flexible aircraft and can be described as

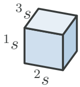
$$s^2 M(m)x(s) + sB(m)x(s) + K(m)x(s) - G(s,v) = u(s),$$

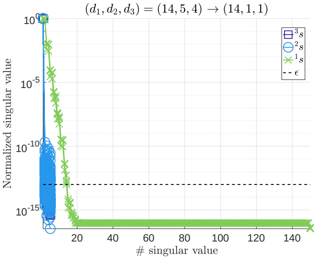
where $M(m), B(m), K(m) \in \mathbb{R}^{n_d \times n_d}$ are the mass, damping, and stiffness matrices, all dependent on the aircraft mass $m \in \mathbb{R}_+$ $(n_d \approx 100)$.

- These matrices are constant for a given flight point but may vary for a mass configuration; then, the generalized aeroelastic forces $G(s,v) \in \mathbb{C}^{n \times n}$ describe the aeroelastic forces exciting the structural dynamic.
- $G(s,v)$ is known at a few sampled frequencies and some true airspeed values, i.e.,

$$G(\imath\omega_i, v_j), \text{ where } i = 1, \cdots, 150, \ \& \ j = 1, \cdots, 10.$$

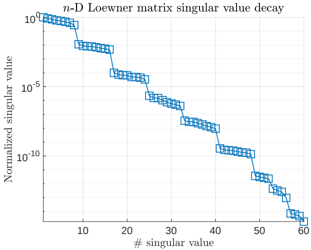
- These values are obtained through dedicated high-fidelity numerical solvers.

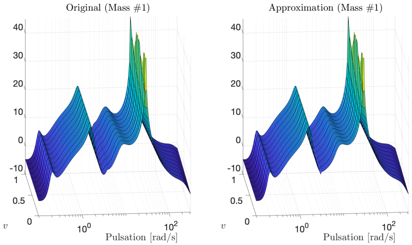




$$\begin{array}{ccccc} {}^1s & \times & {}^2s & \times & {}^3s \\ \imath[10, 35] & \times & [\underline{m}, \overline{m}] & \times & [\underline{v}, \overline{v}] \end{array}$$

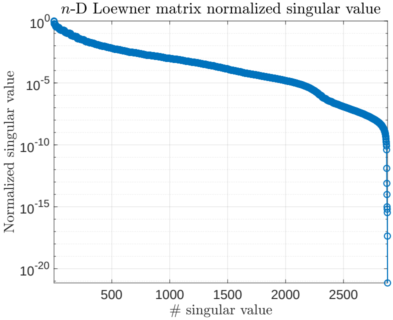
$\mathbf{tab}_3 \in \mathbb{C}^{300 \times 10 \times 10}$

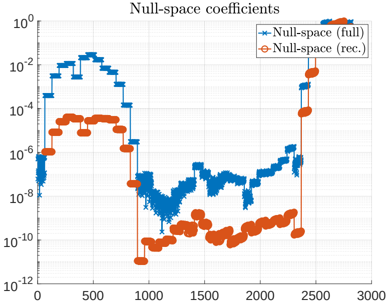




- In the 7D case (6 masses as parameters), the objective is sensitivity analysis and robustness for one single flight point (the airspeed v is not a parameter).

$$\boldsymbol{\Sigma}(s, m_1, \cdots, m_6) : s^2 M(m_1, \cdots, m_6) x(s) + sBx(s) + Kx(s) - G(s) = u(s)$$





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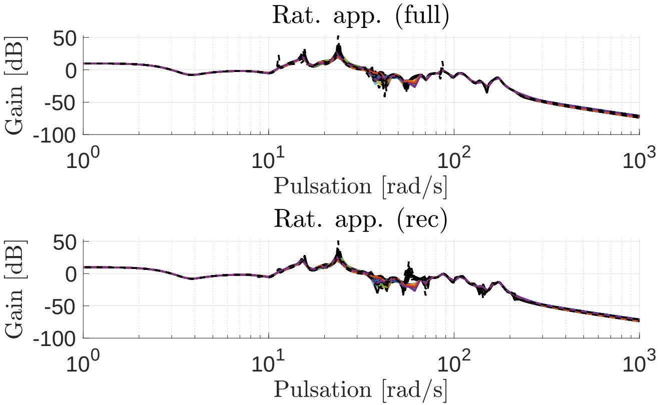
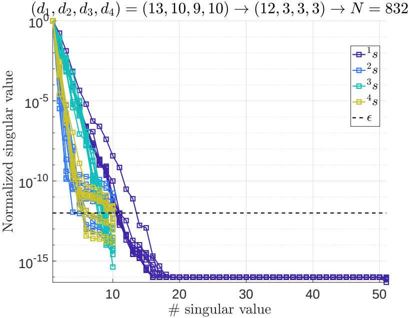


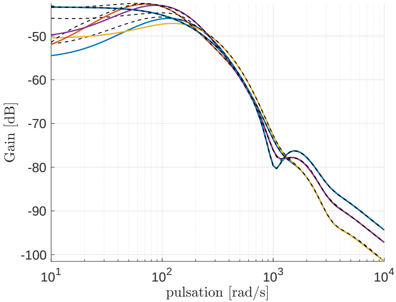
Numerical Examples ⇝ Anemometer (MorWiki[4])

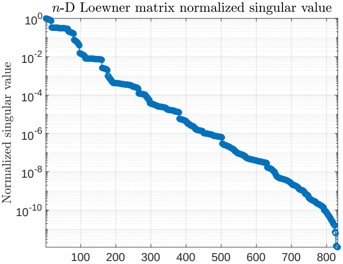
- An anemometer is a flow-sensing device, consisting of a heater & temperature sensors before & after the heater, placed either directly in the flow or in its vicinity.
- The solid model has been generated and meshed in ANSYS; triangular elements have been used for the FEM, with the order of the system $n_d = 29008$.
- The 3 parameters p_0, p_1, p_2 are combinations of the original fluid parameters:

$$H(s, p_0, p_1, p_2) = \mathbf{C}[s(\mathbf{E}_s + p_0\mathbf{E}_f) - (\mathbf{A}_{d,s} + p_1\mathbf{A}_{d,f} + p_2\mathbf{A}_c)]^{-1}\mathbf{B}.$$

[4] Available at: `https://morwiki.mpi-magdeburg.mpg.de/morwiki/index.php/Anemometer`


$(d_1, d_2, d_3, d_4) = (13, 10, 9, 10) \rightarrow (12, 3, 3, 3) \rightarrow N = 832$
Normalized singular value
singular value
^1s
^2s
^3s
^4s
ϵ


Gain [dB]
pulsation [rad/s]


n-D Loewner matrix normalized singular value
Normalized singular value

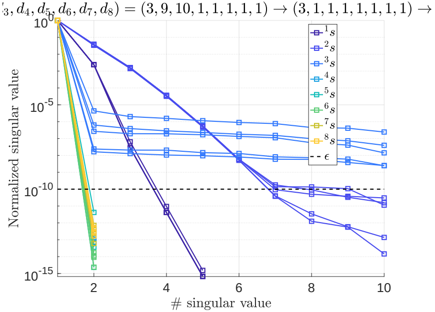
Numerical Examples ⇝ Borehole function (SFU library[5])

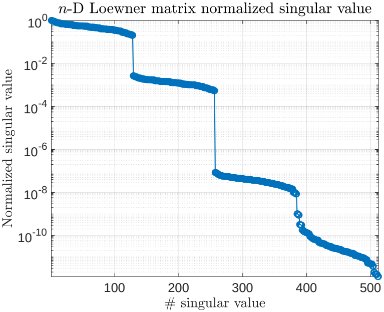
- The Borehole function models water flow through a borehole.
- Its simplicity and quick evaluation make it a commonly used function for testing a wide variety of methods in computer experiments.
- It depends on $n = 8$ parameters and the response is water flow rate, in m^3/yr.

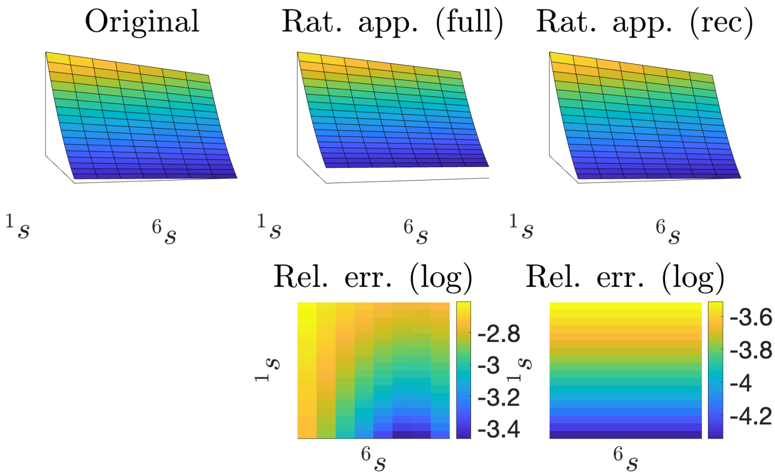
[5] Available at: `https://www.sfu.ca/~ssurjano/borehole.html`

The vector of parameters is $\mathbf{x} = [r_w, r, T_u, H_u, T_l, H_l, L, K_w]$, with the function:

$$f(\mathbf{x}) = \frac{2\pi T_u (H_u - H_l)}{\ln\left(\frac{r}{r_w}\right)\left(1 + \frac{2LT_u}{\ln(r/r_w) r_w^2 K_w}\right) + \frac{T_u}{T_l}}$$



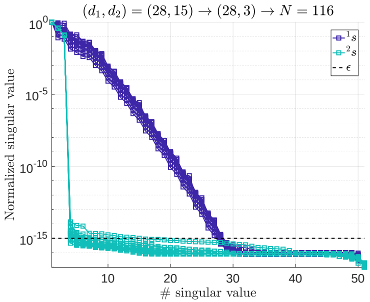


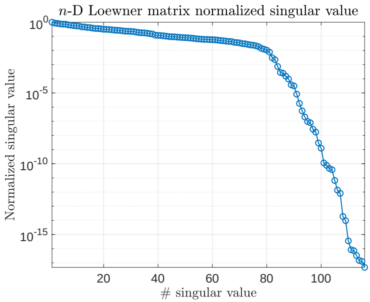

Original
Rat. app. (full)
Rat. app. (rec)
s_1
s_6
s_1
s_6
s_1
s_6
Rel. err. (log)
Rel. err. (log)
s_1
s_6
-2.8
-3
-3.2
-3.4
s_1
s_6
-3.6
-3.8
-4
-4.2

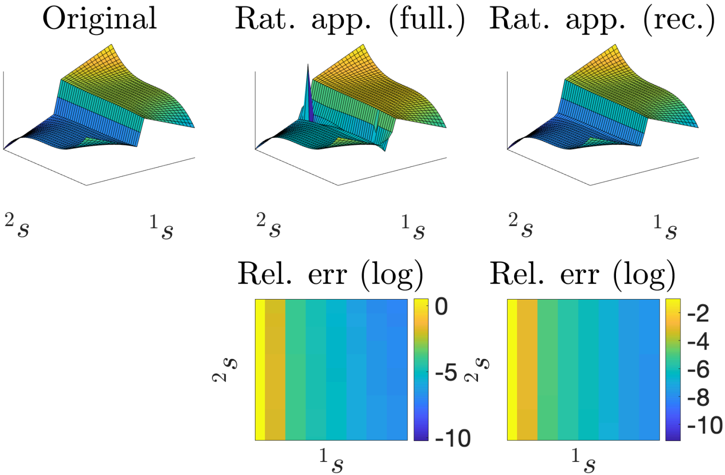
Numerical Examples $\rightsquigarrow$ Non-differentiable functions

1). Saturation-like function:

$$f(x,y) = \mathbf{min}(10|x|,1)\mathbf{sign}(x) + \frac{xy^3}{10}$$

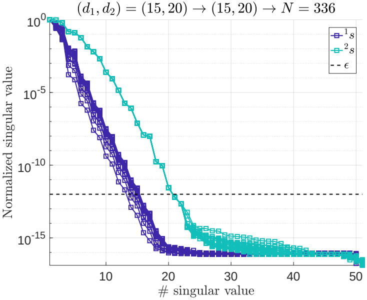


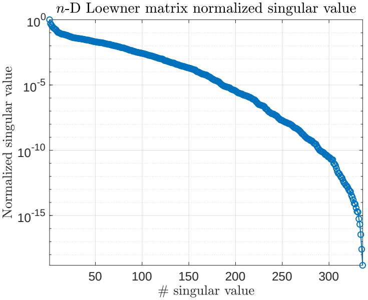


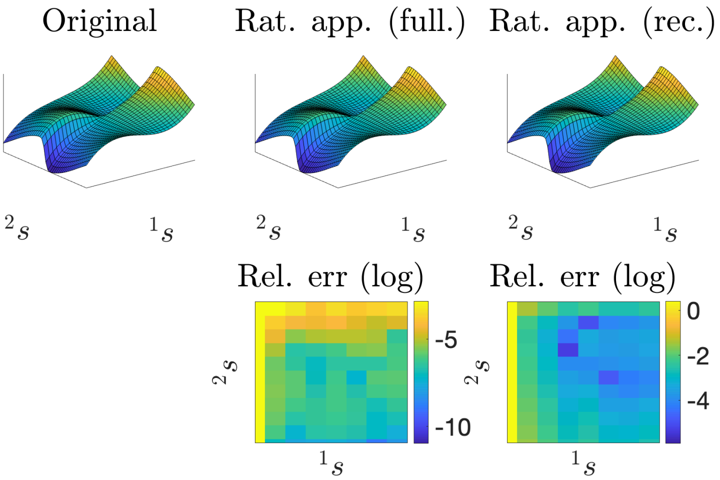

Original
Rat. app. (full.)
Rat. app. (rec.)
s_2
s_1
Rel. err (log)
Rel. err (log)
0
-5
-10
-2
-4
-6
-8
-10

2). Saturation/trigonometric-like function:

$$f(x,y) = \mathbf{atan}(x)\mathbf{cos}(y)\mathbf{tanh}(-10y) + \frac{x^3}{5}$$






Original
Rat. app. (full.)
Rat. app. (rec.)
s_2
s_1
Rel. err (log)
Rel. err (log)
-5
-10
0
-2
-4

Numerical Examples ⇝ Synthetic Parametric Model (MorWiki[6])

[6] Available at: `https://morwiki.mpi-magdeburg.mpg.de/morwiki/index.php/Synthetic_parametric_model`

- We continue with the synthetic model proposed below:

  `https://morwiki.mpi-magdeburg.mpg.de/morwiki/index.php/Synthetic_parametric_model`

- The transfer function is given by:

$$\mathbf{H}(s,p) = \mathbf{C}(s\mathbf{I}_n - (p\mathbf{A}_e + \mathbf{A}_0))^{-1}\mathbf{B},$$

where $n = 100$ and where $\mathbf{C}, \mathbf{A}_e, \mathbf{A}_0, \mathbf{B}$ are matrices of appropriate dimensions.

- We continue with the synthetic model proposed below:

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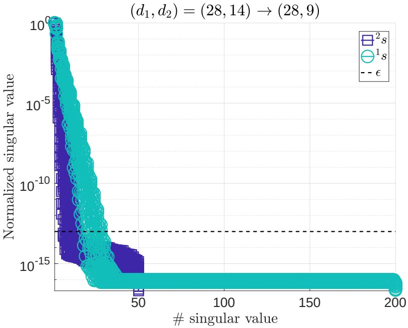
- We use the following sampling points:
  - s variable: 200 points logarithmically spaced between $[10^{0.5}, 10^{3.5}]\imath$;
  - p variable: 50 points logarithmically spaced between $[1/20, 1]$;
- The main problem here is that the parametric model can be viewed as a concatenation of non-parametric models with different (minimal) orders.
- Hence, the SV decay of the 2D Loewner framework is fairly slow and it is not clear where to truncate to achieve optimal fit (and to avoid numerical issues).



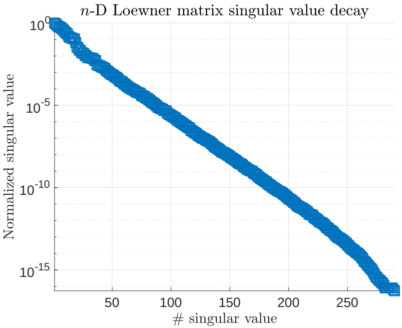


Figure: 2D synthetic model: singular values of 1D (left) and 2D (right) Loewner matrices.

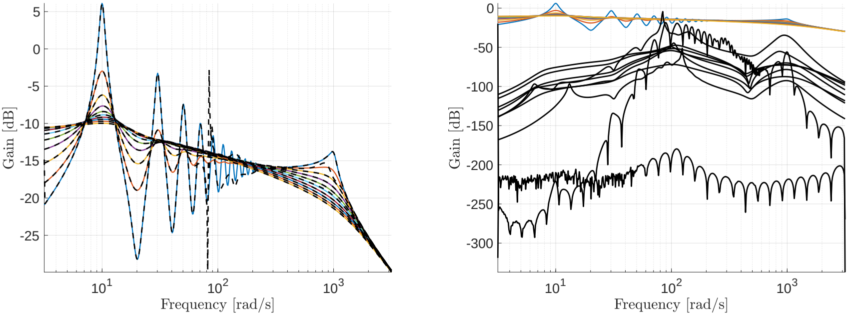


Figure: 2D synthetic model: fitting for various parameter values + approximation errors.

Conclusions

- The contribution for today was submitted to **SIAM Review**, Research Spotlights section.
- It has been available on arXiv since May 2024, and can be accessed at:

  `https://arxiv.org/abs/2405.00495`.
- **Code availability**: everything (code, data, many more test cases) is available online on the following webpage developed by Charles Poussot-Vassal:

  `https://sites.google.com/site/charlespoussotvassal/nd_loew_tcod`.

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- The target audience for this review paper is SIAM CSE community (not only for MOR people, but also SciComp, Tensor, and NonLinEIG people).

Some selected references:

1. A.C. Antoulas, C.A. Beattie, S. Gugercin, *Interpolatory methods for model reduction*, SIAM, CS volume 21, 2020.
2. A.C. Antoulas, I. V. G., C. Poussot-Vassal, The Loewner framework for parametric systems: Taming the curse of dimensionality , arXiv:2405.00495, May 2024.
3. A.C. Antoulas, S. Lefteriu, and A.C. Ionita, A Tutorial Introduction to the Loewner Framework for Model Reduction, in CSE, Model Reduction and Approximation: Theory and Algorithms, SIAM, pp. 335-376 (2017).
4. P. Benner, S. Gugercin, K. Willcox, A Survey of Projection-Based Model Reduction Methods for Parametric Dynamical Systems, SIAM Review, **57**: 483-531, (2015).
5. D, Quero, P. Vuillemin, C. Poussot-Vassal, *A generalized eigenvalue solution to the flutter stability problem with true damping: The p-L method*, Journal of Fluids and Structures, **103**, (2021).
6. A. dos Reis de Souza, C. Poussot-Vassal, P. Vuillemin, and J. Toledo Zucco, *Aircraft flutter suppression: from a parametric model to robust control*, Proceedings of the ECC Conference, Bucharest, June 2023.

Conclusions

- The contribution for today was submitted to **SIAM Review**, Research Spotlights Section.
- It has been available on arXiv since May 2024, and can be accessed at:

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' **Thanks for your attention!**.

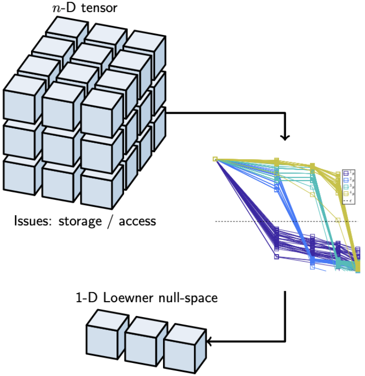
Appendix

Numerical Examples ⇝ Synthetic Toy Model in 20 parameters

- Consider the following rational function in 20 variables: $\mathbf{H}({}^1s, \ldots, {}^{20}s) =$

$$\frac{3 \cdot {}^1s^3 + 4 \cdot {}^8s + {}^{12}s + {}^{13}s \cdot {}^{14}s + {}^{15}s}{{}^1s + {}^2s^2 \cdot {}^3s + {}^4s + {}^5s + {}^6s + {}^7s \cdot {}^8s + {}^9s \cdot {}^{10}s \cdot {}^{11}s + {}^{13}s + {}^{13}s^3 \cdot \pi + {}^{17}s + {}^{18}s \cdot {}^{19}s - {}^{20}s}$$

  with a complexity of $(3, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 3, 1, 1, 1, 1)$.
- This requires us in the best case scenario to construct a gigantic N-D Loewner matrix $\mathbb{L}_n$, with $N = 6,291,456$ columns, and (at least) as many number of rows.



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  with a complexity of $(3,2,1,1,1,1,1,1,1,1,1,1,1,1,1,3,1,1,1,1)$.
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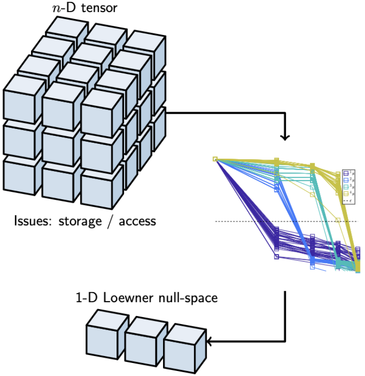

$\rightsquigarrow$ By applying the recursive 1-D null space construction, the barycentric coefficients $\mathbf{c}_n\in\mathbb{C}^N$ are obtained with a computational complexity of 54,263,884 `flops`, computed in 55 mins on a laptop.

$\rightsquigarrow$ Storing such a $N\times N$ n-D Loewner matrix would require $294,912$ GB in double precision).

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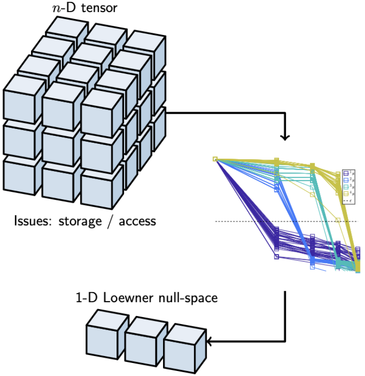

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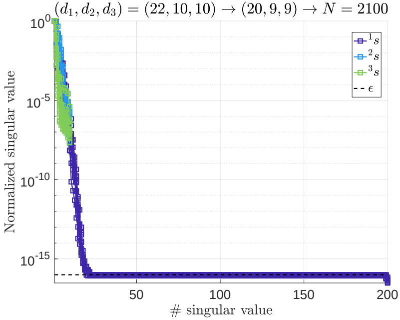
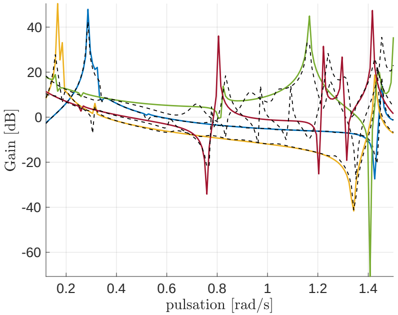
$\rightsquigarrow$ **Thank you for the attention!**

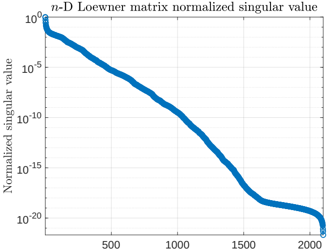
Numerical Examples ⇝ Modified Gyroscope (MorWiki[7])

- The device is a MEMS gyroscope based on the butterfly gyroscope developed at the Imego Institute in Gothenburg, Sweden.
- The model depends on 2 parameters, with the device working in the frequency range $f \in [0.025, 0.25]$MHz; the degrees of freedom are $n_d = 17913$.
- The first is the quantity that is to be sensed, the rotation velocity θ around x-axis.
- The second parameter is the width of the bearing, d.

[7] Available at: https://morwiki.mpi-magdeburg.mpg.de/morwiki/index.php/Modified_Gyroscope

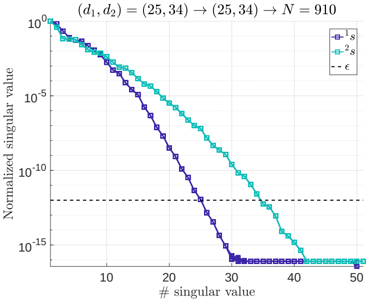

$(d_1, d_2, d_3) = (22, 10, 10) \to (20, 9, 9) \to N = 2100$
Normalized singular value
singular value
^1s
^2s
^3s
ϵ

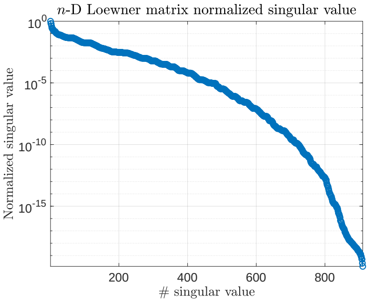

Gain [dB]
pulsation [rad/s]

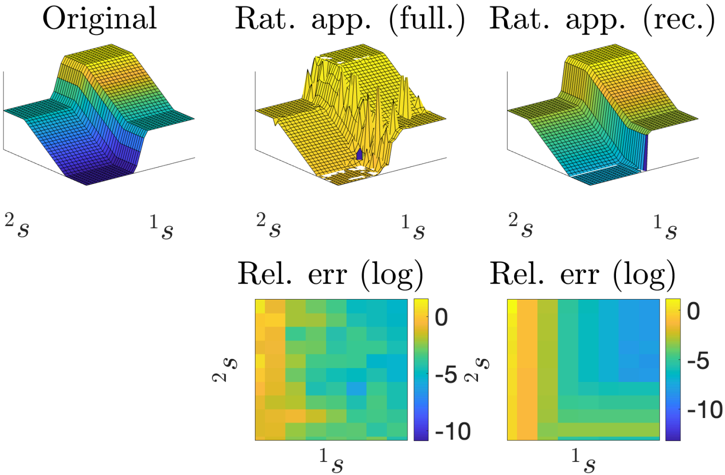

n-D Loewner matrix normalized singular value
Normalized singular value

Saturation-like function:

$$f(x,y) = \mathbf{min}(5|x|,1)\mathbf{sign}(x) + \mathbf{min}(|y|,1)\mathbf{sign}(y)$$






Original
Rat. app. (full.)
Rat. app. (rec.)
s_2
s_1
Rel. err (log)
Rel. err (log)
0
-5
-10

The first proposed innovation

- We construct data-based realizations for any number of parameters n.
- This is useful to circumvent the convoluted multi-parameter barycentric form when evaluating the fitted function, and instead use:

$$\tilde{\mathbf{H}}({}^1s, \cdots, {}^ns) = \mathbf{W}\mathbf{\Phi}({}^1s, {}^2s, \cdots, {}^ns)^{-1}\mathbf{G}, \tag{1}$$

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  and comes as an extension of the theoretical works from the early 2010's.
- A $(2\ell+\kappa-1)$-th order realization $(\mathbf{G},\mathbf{\Phi},\mathbf{W})$ of the multivariate function in barycentric form $\tilde{\mathbf{H}}$, is given by:

$$\mathbf{\Phi}({}^1s,\cdots,{}^ns)=\left[\begin{array}{c|c|c}\mathbf{\Gamma}(1:\kappa-1,:) & \mathbf{0}_{\kappa-1,\ell-1} & \mathbf{0}_{\kappa-1,\ell}\\ \hline \mathbb{A}^{\mathrm{Lag}} & \mathbf{\Delta}(1:\ell-1,:)^\top & \mathbf{0}_{\ell,\ell}\\ \hline \mathbb{B}^{\mathrm{Lag}} & \mathbf{0}_{\ell,\ell-1} & \mathbf{\Delta}^\top\end{array}\right]\in\mathbb{C}^{m\times m}[{}^1s,\cdots,{}^ns],$$

$$\mathbf{G}=\left[\begin{array}{c}\mathbf{0}_{\kappa-1,1}\\ \hline \mathbf{\Delta}(\ell,:)^\top\\ \hline \mathbf{0}_{\ell,1}\end{array}\right]\in\mathbb{C}^{m\times 1}\ \text{and}\ \mathbf{W}=\left[\begin{array}{c|c|c}\mathbf{0}_{1,\kappa} & \mathbf{0}_{1,\ell-1} & -\mathbf{e}_\ell^\top\end{array}\right]\in\mathbb{C}^{1\times m}$$

The second proposed innovation

- The n-D Loewner matrix $\mathbb{L}_n$ is the solution of the coupled Sylvester equations:

$$\begin{cases} \mathbf{M}_n\mathbb{X}_1 - \mathbb{X}_1\mathbf{\Lambda}_n & = & \mathbb{V}_n\mathbf{R}_n - \mathbf{L}_n\mathbb{W}_n, \\ \mathbf{M}_{n-1}\mathbb{X}_2 - \mathbb{X}_2\mathbf{\Lambda}_{n-1} & = & \mathbb{X}_1, \\ & \cdots & \\ \mathbf{M}_2\mathbb{X}_{n-1} - \mathbb{X}_{n-1}\mathbf{\Lambda}_2 & = & \mathbb{X}_{n-2}, \\ \mathbf{M}_1\mathbb{L}_n - \mathbb{L}_n\mathbf{\Lambda}_1 & = & \mathbb{X}_{n-1}, \end{cases}$$

or as 1 generalized Sylvester equation with 2^n left terms; e.g., for $n=2$, write:

$$\mathbf{M}_2\mathbf{M}_1\mathbb{L}_2 - \mathbf{M}_2\mathbb{L}_2\mathbf{\Lambda}_1 - \mathbf{M}_1\mathbb{L}_2\mathbf{\Lambda}_2 + \mathbb{L}_2\mathbf{\Lambda}_1\mathbf{\Lambda}_2 = \mathbb{V}_2\mathbf{R}_2 - \mathbf{L}_2\mathbb{W}_2.$$

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- The null space of n-D Loewner matrices gives us the barycentric weights.
- We show the null space of the n-dimensional Loewner matrix $\mathbb{L}_n$ can be **recursively computed** using a sequence of 1-dimensional Loewner matrices $\mathbb{L}_1$, leading to a drastic computational burden reduction.

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- **Many more details were provided in the talk by Thanos Antoulas.**

Require: **tab**$_n$

1: Check that interpolation points are disjoints.

2: Compute $d_j = \max_k \mathbf{rank}\ \mathbb{L}_1^{(k)}$, the order along variable ${}^j s$ (k being all possible combinations for frozen the variables $\{{}^1 s, \cdots, {}^{k-1}s, {}^{k+1}s, \cdots, {}^n s\}$).

3: Construct data set, a sub-selection $P_c^{(n)}$ where $(k_1, k_2 \ldots, k_n) = (d_1, d_2, \ldots, d_n) + 1$; and $P_r^{(n)}$ where $(q_1, q_2 \ldots, q_n)$ gather the rest of the data.

4: Compute $\mathbf{c}_n$, the n-D Loewner matrix null space.

5: Construct $\mathbb{A}^{\mathrm{Lag}}$, $\mathbb{B}^{\mathrm{Lag}}$, $\boldsymbol{\Gamma}$ and $\boldsymbol{\Delta}$ with any left/right separation.

6: Construct multivariate realization.

Ensure: $\hat{\mathbf{H}}({}^1 s, \ldots, {}^n s) = \mathbf{W}\boldsymbol{\Phi}({}^1 s, {}^2 s, \ldots, {}^n s)^{-1}\mathbf{G}$ interpolates $\mathbf{H}({}^1 s, {}^2 s, \ldots, {}^n s)$ along $P_c^{(n)}$.

Require: **tab**$_n$ and tolerance $\texttt{tol} > 0$

Check that interpolation points are disjoints.

while $\texttt{error} > \texttt{tol}$ **do**

  Search the point indexes with maximal $\texttt{error}$ (first iteration: pick any set).

  Add points in $P_c^{(n)}$ and put the remaining ones in $P_r^{(n)}$, obtain data set.

  Compute $\mathbf{c}_n$, the n-D Loewner matrix null space.

  Construct $\mathbb{A}^{\mathrm{Lag}}$, $\mathbb{B}^{\mathrm{Lag}}$, $\mathbf{\Gamma}$ and $\mathbf{\Delta}$ with any left/right separation.

  Construct multivariate realization.

  Evaluate $\texttt{error} = \max ||\widehat{\textbf{tab}_n} - \textbf{tab}_n||$ where $\widehat{\textbf{tab}_n}$ is the evaluation of $\hat{\mathbf{H}}({}^1s, \ldots, {}^ns)$ along the support points.

end while

Ensure: $\hat{\mathbf{H}}({}^1s, \ldots, {}^ns) = \mathbf{W}\mathbf{\Phi}({}^1s, {}^2s, \ldots, {}^ns)^{-1}\mathbf{G}$ interpolates $\mathbf{H}({}^1s, {}^2s, \ldots, {}^ns)$ along $P_c^{(n)}$.