The parametric Loewner Framework & the Curse-of-Dimensionality

Thanos Antoulas

Joint work with: Ion-Victor Goşea & Charles Poussot-Vassal

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Problem Set-up

- **Data**: $s_i \in \mathbb{C}$, $\phi_i \in \mathbb{C}$, $i = 1, \cdots, N$.
- **Find**: rational function $\mathbf{H}(s) = \frac{\mathbf{n}(s)}{\mathbf{d}(s)}$,

  s.t. $\mathbf{H}(s_i) = \phi_i$, $i = 1, \cdots, N$.

- **Omitted**: Various generalizations of the Loewner framework to MIMO systems, nonlinear systems, etc.

$\Downarrow$

Solution approaches:

▲ $\boxed{\textbf{Barycentric framework (AAA)}}$

▲ Loewner pencil

$\Rightarrow$ $\rightarrow \begin{cases} \text{Single-variable functions} \sim \text{linear systems} \\ \text{Multi-variate functions} \sim \textbf{parametrized} \text{ linear systems} \end{cases}$

Lagrange basis, the Loewner matrix and rational interpolation

- **Lagrange basis**: Given $\lambda_i \in \mathbb{C}$, $\mathbf{q}_i(s) = \Pi_{i' \neq i}(s - \lambda_{i'})$, $i = 1, \cdots, n+1$.
- For given constants α_i, $\mathbf{w}_i$, $i = 1, \cdots, n+1$, consider: $\boxed{\sum_{i=1}^{n+1} \alpha_i \frac{\mathbf{g} - \mathbf{w}_i}{s - \lambda_i} = 0, \quad \alpha_i \neq 0}$
- Solving for $\mathbf{g}$ we obtain

$$\boxed{\mathbf{g}(s) = \frac{\sum_{i=1}^{n+1} \frac{\alpha_i \mathbf{w}_i}{s - \lambda_i}}{\sum_{i=1}^{n+1} \frac{\alpha_i}{s - \lambda_i}}} = \frac{\sum_i \alpha_i \, \mathbf{w}_i \, \mathbf{q}_i(s)}{\sum_i \alpha_i \, \mathbf{q}_i(s)} \Rightarrow \mathbf{g}(\lambda_i) = \mathbf{w}_i, \ \forall \alpha_i,$$

This is the **barycentric Lagrange interpolation** formula.

The free parameters α_i, can be specified so that *additional* interpolation conditions are satisfied:

$\mathbf{g}(\mu_j) = \mathbf{v}_j, \ j = 1, \cdots, r$. For this to hold $\mathbb{L}\,\mathbf{c} = 0$, where

$$\mathbb{L} = \begin{bmatrix} \frac{\mathbf{v}_1 - \mathbf{w}_1}{\mu_1 - \lambda_1} & \cdots & \frac{\mathbf{v}_1 - \mathbf{w}_{n+1}}{\mu_1 - \lambda_{n+1}} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_r - \mathbf{w}_1}{\mu_r - \lambda_1} & \cdots & \frac{\mathbf{v}_r - \mathbf{w}_{n+1}}{\mu_r - \lambda_{n+1}} \end{bmatrix} \in \mathbb{C}^{r \times (n+1)}, \quad \mathbf{c} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_{n+1} \end{bmatrix} \in \mathbb{C}^{n+1}.$$

$\mathbb{L}$: **Loewner matrix** with **row array** $(\mu_j, \mathbf{v}_j)$, $j = 1, \cdots, r$, and **column array** $(\lambda_i, \mathbf{w}_i)$, $i = 1, \cdots, n+1$. The coefficients α_i are called **barycentric weights**.

Realization of the barycentric representation

Consider a rational function $\mathbf{H}(s)$, of the degree n. Let the Lagrange monomials be $\mathbf{x}_j = s - \lambda_j$, $j = 1, \cdots, n+1$, $\lambda_j \in \mathbb{C}$. Define:

• **pseudo-companion form** matrix of dimension $n \times (n+1)$:

$$\mathbb{X}(s) = \begin{bmatrix} \mathbf{x}_1 & -\mathbf{x}_2 & 0 & \cdots & 0 \\ \mathbf{x}_1 & 0 & -\mathbf{x}_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{x}_1 & 0 & \cdots & 0 & -\mathbf{x}_{n+1} \end{bmatrix} \in \mathbb{C}^{n\times(n+1)}[s],$$

• **coefficient matrices**: $\mathbb{A} = [\; a_1 \quad a_2 \quad \cdots \quad a_{n+1} \;]$, $\mathbb{B} = [\; b_1 \quad b_2 \quad \cdots \quad b_{n+1} \;] \in \mathbb{C}^{1\times(n+1)}$.

• $\mathbb{A}$: contains the barycentric weights and $\mathbb{B}$: the barycentric weights times the associated values of $\mathbf{H}$.

Theorem. Putting these quantities together we obtain a realization of $\mathbf{H}(s)$:

$$\boldsymbol{\Phi}(s) = \underbrace{\left[\begin{array}{c|c} \mathbb{X}(s) & \mathbf{0} \\ \hline \mathbb{A} & \mathbf{0} \\ \hline \mathbb{B} & 1 \end{array}\right]}_{(n+2)\times(n+2)}, \quad \mathbf{G} = \left[\begin{array}{c} \mathbf{0} \\ \hline 1 \\ \hline \mathbf{0} \end{array}\right], \quad \mathbf{W} = [\,\mathbf{0} \mid \mathbf{0} \mid -\mathbf{1}\,] \;\Rightarrow\; \boxed{\mathbf{H}(s) = \mathbf{W}\,\boldsymbol{\Phi}(s)^{-1}\,\mathbf{G}}.$$

The realization $(\mathbf{W}, \boldsymbol{\Phi}, \mathbf{G})$ has dimension $n+2$, and is both R-controllable and R-observable, i.e. $[\boldsymbol{\Phi},\ \mathbf{G}]$ and $\begin{bmatrix} \mathbf{H} \\ \boldsymbol{\Phi} \end{bmatrix}$, have full rank $n+2$, for all $s \in \mathbb{C}$.

The Loewner matrix - 2D interpolation

- $\mathcal{P}_{n,m}$: space of polynomials in two indeterminates, s and t, so that degree with respect to s is at most n and degree with respect to t is at most m $\Rightarrow$ $\dim \mathcal{P}_{n,m} = (n+1)(m+1)$.
- **Lagrange basis**:

$$\mathbf{q}_{i,j}(s,t) = \prod_{i' \neq i}(s-\lambda_{i'}) \prod_{j' \neq j}(t-\pi_{j'}), \;\; i=1,\cdots,n+1, \;\; j=1,\cdots,m+1.$$

- Let $\mathbf{g}(s,t)$ satisfy: $\boxed{\sum_{i=1}^{n+1}\sum_{j=1}^{m+1} \alpha_{i,j} \frac{\mathbf{g}-\mathbf{w}_{i,j}}{(s-\lambda_i)(t-\pi_j)} = 0, \;\; \alpha_{i,j} \neq 0}$ $\Rightarrow$

$$\boxed{\mathbf{g}(s,t) = \frac{\sum_{i=1}^{n+1}\sum_{j=1}^{m+1} \frac{\alpha_{i,j}\mathbf{w}_{i,j}}{(s-\lambda_i)(t-\pi_j)}}{\sum_{i=1}^{n+1}\sum_{j=1}^{m+1} \frac{\alpha_{i,j}}{(s-\lambda_i)(t-\pi_j)}}} = \frac{\sum_{i=1}^{n+1}\sum_{j=1}^{m+1} \alpha_{i,j}\mathbf{w}_{i,j}\,\mathbf{q}_{i,j}(s,t)}{\sum_{i=1}^{n+1}\sum_{j=1}^{m+1} \alpha_{i,j}\,\mathbf{q}_{i,j}(s,t)}.$$

This is the **two-variable barycentric representation formula**. It follows that $\mathbf{g}$ satisfies the interpolation conditions $\mathbf{g}(\lambda_i,\pi_j) = \mathbf{w}_{i,j}$.

- The weights $\alpha_{i,j}$ can be determined so that $\mathbf{g}$ satisfies additional interpolation conditions:

$$\mathbf{g}(\mu_i,\nu_j) = \mathbf{v}_{i,j}, \;\; i=1,\cdots,p+1, \;\; j=1,\cdots,r+1,$$

where $(\mu_j,\nu_j;\mathbf{v}_{i,j})$ are given.

2D Loewner matrix

- Consider the arrays $\begin{cases} P_c = \{(\lambda_j, \pi_i; \mathbf{w}_{j,i}) : \; i = 1, \cdots, n', \; j = 1, \cdots, m' \\ P_r = \{(\mu_l, \nu_k; \mathbf{v}_{l,k}) : \; k = 1, \cdots, p', \; l = 1, \cdots, r' \end{cases}$.
- The measurements can also be depicted as follows:

$$\left.\begin{array}{c}\textbf{tableau of}\\ \textbf{values}\end{array}\right\} = \left[\begin{array}{c|c} \textbf{right} & \\ \hline & \textbf{left} \end{array}\right] = \begin{array}{c|ccc|ccc} \phi_{i,j} & \pi_1 & \cdots & \pi_{m'} & \nu_1 & \cdots & \nu_{r'} \\ \hline \lambda_1 & \mathbf{w}_{1,1} & \cdots & \mathbf{w}_{1,m'} & \phi_{1,m'+1} & \cdots & \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \\ \lambda_{n'} & \mathbf{w}_{n',1} & \cdots & \mathbf{w}_{n',m'} & \phi_{n',m'+1} & \cdots & \\ \hline \mu_1 & \phi_{n'+1,1} & \cdots & \phi_{n'+1,m'} & \mathbf{v}_{1,1} & \cdots & \mathbf{v}_{1,M} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \mu_{p'} & & & & \mathbf{v}_{N,1} & \cdots & \mathbf{v}_{N,M} \end{array}$$

- The **2D Loewner matrix** is:

$$\mathbb{L}_{2D} = \begin{bmatrix} \frac{\mathbf{v}_{1,1}-\mathbf{w}_{1,1}}{(\mu_1-\lambda_1)(\nu_1-\pi_1)} & \frac{\mathbf{v}_{1,1}-\mathbf{w}_{1,2}}{(\mu_1-\lambda_1)(\nu_1-\pi_2)} & \cdots & \frac{\mathbf{v}_{1,1}-\mathbf{w}_{n',m'}}{(\mu_1-\lambda_{n'})(\nu_1-\pi_{m'})} \\ \frac{\mathbf{v}_{1,2}-\mathbf{w}_{1,1}}{(\mu_1-\lambda_1)(\nu_2-\pi_1)} & \frac{\mathbf{v}_{1,2}-\mathbf{w}_{1,2}}{(\mu_1-\lambda_1)(\nu_2-\pi_2)} & \cdots & \frac{\mathbf{v}_{1,2}-\mathbf{w}_{n',m'}}{(\mu_1-\lambda_{n'})(\nu_2-\pi_{m'})} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_{r',p'}-\mathbf{w}_{1,1}}{(\mu'_r-\lambda_1)(\nu'_p-\pi_1)} & \frac{\mathbf{v}_{r',p'}-\mathbf{w}_{1,2}}{(\mu'_r-\lambda_1)(\nu'_p-\pi_2)} & \cdots & \frac{\mathbf{v}_{r',p'}-\mathbf{w}_{n',m'}}{(\mu'_r-\lambda_{n'})(\nu'_p-\pi_{m'})} \end{bmatrix} \in \mathbb{C}^{p'r' \times n'm'}.$$

- Thus $\mathbf{g}$ satisfies the additional interpolation constraints given by P_r, if $\mathbb{L}_{2D}\,\mathbf{c} = 0$.

Multivariate Loewner matrices and generalized Sylvester equations

The definition of $\mathbb{L}_{\rm ND}$ follows by means of Sylvester equations.

- $N=2$. Introduce the diagonal matrices $\mathbf{\Lambda},\mathbf{\Pi}\in\mathbb{C}^{k_r q_r\times k_r q_r}$ and $\mathbf{M},\mathbf{N}\in\mathbb{C}^{k_\ell q_\ell\times k_\ell q_\ell}$:

$$\begin{aligned}\mathbf{\Lambda} &= \operatorname{diag}(\lambda_1,\cdots,\lambda_{k_r})\otimes\mathbf{I}_{q_r},\;\; \mathbf{\Pi}=\mathbf{I}_{k_r}\otimes\operatorname{diag}(\pi_1,\cdots,\pi_{q_r}),\\ \mathbf{M} &= \operatorname{diag}(\mu_1,\cdots,\mu_{k_\ell})\otimes\mathbf{I}_{q_\ell},\;\; \mathbf{N}=\mathbf{I}_{k_\ell}\otimes\operatorname{diag}(\nu_1,\cdots,\nu_{q_\ell}).\end{aligned}$$

Additionally, let $\mathbf{R}\in\mathbb{C}^{1\times k_r q_r}$ and $\mathbf{L}\in\mathbb{C}^{k_\ell q_\ell\times 1}$ be vectors of ones.

Lemma. The 2D Loewner matrix $\mathbb{L}_{\rm 2D}$ satisfies the **generalized Sylvester equation**:

$$\mathbf{N}\,\mathbf{M}\,\mathbb{L}_{\rm 2D}-\mathbf{N}\,\mathbb{L}_{\rm 2D}\,\mathbf{\Lambda}-\mathbf{M}\,\mathbb{L}_{\rm 2D}\,\mathbf{\Pi}+\mathbb{L}_{\rm 2D}\,\mathbf{\Lambda}\,\mathbf{\Pi}=\mathbb{V}\,\mathbf{R}-\mathbf{L}\,\mathbb{W} \qquad (*)$$

Corollary. $\mathbb{L}_{\rm 2D}$, satsfies two standard coupled Sylvester equations:

$$\mathbf{N}\,\mathbb{X}-\mathbb{X}\,\mathbf{\Pi}=\mathbb{V}\,\mathbf{R}-\mathbf{L}\,\mathbb{W},\;\; \mathbf{M}\,\mathbb{L}_{\rm 2D}-\mathbb{L}_{\rm 2D}\,\mathbf{\Lambda}=\mathbb{X}.$$

When $\mathbb{X}$ is eliminated the generalized Sylvester equation $(*)$ results.

Loewner matrices $\mathbb{L}_{3\mathrm{D}}$

We introduce the **right data** $\mathbf{\Lambda}, \mathbf{\Pi}, \mathbf{\Theta} \in \mathbb{C}^{k_r q_r v_r \times k_r q_r v_r}$, and the **left data** $\mathbf{M}, \mathbf{N}, \mathbf{Z} \in \mathbb{C}^{k_\ell q_\ell v_\ell \times k_\ell q_\ell v_\ell}$:

$$\begin{aligned} \mathbf{\Lambda} &= \mathrm{diag}(\lambda_1, \ldots, \lambda_{k_r}) \otimes \mathbf{I}_{q_r} \otimes \mathbf{I}_{v_r}, & \mathbf{M} &= \mathrm{diag}(\mu_1, \ldots, \mu_{k_\ell}) \otimes \mathbf{I}_{q_\ell} \otimes \mathbf{I}_{v_\ell}, \\ \mathbf{\Pi} &= \mathbf{I}_{k_r} \otimes \mathrm{diag}(\pi_1, \ldots, \pi_{q_r}) \otimes \mathbf{I}_{v_r}, & \mathbf{N} &= \mathbf{I}_{k_\ell} \otimes \mathrm{diag}(\nu_1, \ldots, \nu_{q_\ell}) \otimes \mathbf{I}_{v_\ell}, \\ \mathbf{\Theta} &= \mathbf{I}_{k_r} \otimes \mathbf{I}_{q_r} \otimes \mathrm{diag}(\theta_1, \ldots, \theta_{v_r}), & \mathbf{Z} &= \mathbf{I}_{k_\ell} \otimes \mathbf{I}_{q_\ell} \otimes \mathrm{diag}(\zeta_1, \ldots, \zeta_{v_\ell}). \end{aligned}$$

Let $\mathbf{R} \in \mathbb{C}^{1 \times k_r q_r v_r}$ and $\mathbf{L} \in \mathbb{C}^{k_\ell q_\ell v_\ell \times 1}$ be vectors of ones.

Lemma. The 3D Loewner matrix satisfies the **generalized Sylvester** equation:

$$\mathbf{Z}\,(\mathbf{N}\,\mathbf{M}\,\mathbb{L}_{3\mathrm{D}} - \mathbf{N}\,\mathbb{L}_{3\mathrm{D}}\,\mathbf{\Lambda} - \mathbf{M}\,\mathbb{L}_{3\mathrm{D}}\,\mathbf{\Pi} + \mathbb{L}_{3\mathrm{D}}\,\mathbf{\Lambda}\,\mathbf{\Pi}) - (\mathbf{N}\,\mathbf{M}\,\mathbb{L}_{3\mathrm{D}} - \mathbf{N}\,\mathbb{L}_{3\mathrm{D}}\,\mathbf{\Lambda} - \mathbf{M}\,\mathbb{L}_{3\mathrm{D}}\,\mathbf{\Pi} + \mathbb{L}_{3\mathrm{D}}\,\mathbf{\Lambda}\mathbf{\Pi})\,\mathbf{\Theta}$$
$$= \mathbb{V}\,\mathbf{R} - \mathbf{L}\,\mathbb{W}.$$

Corollary. $\mathbb{L}_{\mathrm{ND}}$ satisfies a sequence of N coupled Sylvester equations:

$$\begin{aligned} &1\mathrm{D}: && \mathbb{L}_{1\mathrm{D}}\,\mathbf{\Lambda} - \mathbf{M}\,\mathbb{L}_{1\mathrm{D}} = \mathbb{V}\,\mathbf{R} - \mathbf{L}\,\mathbb{W}, \\ &2\mathrm{D}: && \mathbf{N}\,\mathbb{X} - \mathbb{X}\,\mathbf{\Pi} = \mathbb{V}\,\mathbf{R} - \mathbf{L}\,\mathbb{W}, \quad \mathbf{M}\,\mathbb{L}_{2\mathrm{D}} - \mathbb{L}_{2\mathrm{D}}\,\mathbf{\Lambda} = \mathbb{X}, \\ &3\mathrm{D}: && \mathbf{Z}\,\mathbb{X}_1 - \mathbb{X}_1\,\mathbf{\Theta} = \mathbb{V}\,\mathbf{R} - \mathbf{L}\,\mathbb{W}, \quad \mathbf{N}\,\mathbb{X}_2 - \mathbb{X}_2\,\mathbf{\Pi} = \mathbb{X}_1, \quad \mathbf{M}\,\mathbb{L}_{3\mathrm{D}} - \mathbb{L}_{3\mathrm{D}}\,\mathbf{\Lambda} = \mathbb{X}_2, \\ & && \vdots \end{aligned}$$

Realization of mutlivariate rational functions

Consider a rational function $\mathbf{H}$ in N variables: ${}^{i}s$, $i = 1 \cdots, N$. Let the degree of $\mathbf{H}$ in each of these variables be n_i, $i = 1, \cdots, N$, and the Lagrange monomials in the variable ${}^{i}s$, be ${}^{i}\mathbf{x}_j = {}^{i}s - {}^{i}\lambda_j$, $j = 1, \cdots, n_i + 1$, ${}^{i}\lambda_j \in \mathbb{C}$. Associated with the i^{th} variable, we define the **pseudo-companion form** matrix:

$$
{}^{i}\mathbb{X} = \left[\begin{array}{ccccc}
{}^{i}\mathbf{x}_1 & -{}^{i}\mathbf{x}_2 & 0 & \cdots & 0 \\
{}^{i}\mathbf{x}_1 & 0 & -{}^{i}\mathbf{x}_3 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
{}^{i}\mathbf{x}_1 & 0 & \cdots & 0 & -{}^{i}\mathbf{x}_{n_i+1} \\
{}^{i}\epsilon_1 & {}^{i}\epsilon_2 & \cdots & {}^{i}\epsilon_{n_i} & {}^{i}\epsilon_{n_i+1}
\end{array}\right] \in \mathbb{C}^{(n_i+1)\times(n_i+1)}[{}^{i}s].
$$

The constants ${}^{i}\epsilon_j$, $j = 1, \cdots, n_i + 1$, are chosen so that ${}^{i}\mathbb{X}$ is **unimodular**.

Next, the variables are split into left (row) and right (column) variables. For simplicity, assume that ${}^{1}s, \cdots, {}^{k}s$ are the right variables and ${}^{k+1}s, \cdots, {}^{N}s$ are the left variables.

Define two **Kronecker products**: size $\kappa \times \kappa$ $\left\{\boldsymbol{\Gamma} = {}^{1}\mathbb{X} \otimes \cdots \otimes {}^{k}\mathbb{X}\right\}$ and $\left\{\boldsymbol{\Delta} = {}^{k+1}\mathbb{X} \otimes \cdots \otimes {}^{N}\mathbb{X}\right\}$ size $\ell \times \ell$, where $\kappa = \Pi_{i=1}^{k}(n_i + 1)$, and $\ell = \Pi_{i=k+1}^{N}(n_i + 1)$. These matrices are unimodular.

Multi-row/multi-column indices and the coefficient matrices. Each column of $\boldsymbol{\Gamma}$ and each column of $\boldsymbol{\Delta}$ defines a unique multi-index I_q, J_r. We will refer to these indices as **row-** and **column-multi-indices**:

$I_q = [i_{k+1}^{q}, i_{k+2}^{q}, \cdots, i_{N}^{q}]$, $J_r = [j_1^{r}, j_2^{r}, \cdots, j_k^{r}]$, $q = 1, \cdots, \ell$, $r = 1, \cdots, \kappa$.

Each multi-index I_q (J_r) contains the indices of the Lagrange monomials involved in the q^{th} (r^{th}) column of $\boldsymbol{\Delta}$ ($\boldsymbol{\Gamma}$), respectively.

Remark. The **ordering** of these multi-indices is imposed by the ordering of the associated Kronecker products.

The coefficient matrices are:

$$\mathbb{A} = \begin{bmatrix} a_{I_1,J_1} & a_{I_1,J_2} & \cdots & a_{I_1,J_\kappa} \\ a_{I_2,J_1} & a_{I_2,J_2} & \cdots & a_{I_2,J_\kappa} \\ \vdots & \vdots & \ddots & \vdots \\ a_{I_\ell,J_1} & a_{I_\ell,J_2} & \cdots & a_{I_\ell,J_\kappa} \end{bmatrix}, \ \mathbb{B} = \begin{bmatrix} b_{I_1,J_1} & b_{I_1,J_2} & \cdots & b_{I_1,J_\kappa} \\ b_{I_2,J_1} & b_{I_2,J_2} & \cdots & b_{I_2,J_\kappa} \\ \vdots & \vdots & \ddots & \vdots \\ b_{I_\ell,J_1} & b_{I_\ell,J_2} & \cdots & b_{I_\ell,J_\kappa} \end{bmatrix} \in \mathbb{C}^{\ell\times\kappa}.$$

$\mathbb{A}$: contains the appropriately arranged barycentric weights of $\mathbf{H}$.

$\mathbb{B}$: contains the product of the barycentric weights times the associated values of $\mathbf{H}$.

Theorem. Putting these quantities together we obtain a realization of $\mathbf{H}({}^1s,\cdots,{}^Ns)$:

$$\boldsymbol{\Phi} = \left[\begin{array}{c|c|c} \boldsymbol{\Gamma}(1:\kappa-1,:) & \mathbf{0}_{\kappa-1,\ell-1} & \mathbf{0}_{\kappa-1,\ell} \\ \hline \mathbb{A} & \boldsymbol{\Delta}(1:\ell-1,:)^T & \mathbf{0}_{\ell,\ell} \\ \hline \mathbb{B} & \mathbf{0}_{\ell,\ell-1} & (\boldsymbol{\Delta})^T \end{array}\right], \ \mathbf{G} = \left[\begin{array}{c} \mathbf{0} \\ \hline \boldsymbol{\Delta}(\ell,:)^T \\ \hline \mathbf{0} \end{array}\right], \ \mathbf{W} = [\,\mathbf{0}_{1,\kappa} \mid \mathbf{0}_{1,\ell-1} \mid -\mathbf{e}_\ell^T\,],$$

where $\mathbf{e}_r$ denotes the r^{th} unit vector. Then: $\boxed{\mathbf{H}({}^1s,\cdots,{}^Ns) = \mathbf{W}\,\boldsymbol{\Phi}({}^1s,\cdots,{}^Ns)^{-1}\,\mathbf{G}}$.

$(\mathbf{W},\boldsymbol{\Phi},\mathbf{G})$ has dimension $n = \kappa + 2\ell - 1$, and is both R-controllable and R-observable, i.e.

$$[\boldsymbol{\Phi},\ \mathbf{G}] \ \text{ and } \ \begin{bmatrix} \mathbf{H} \\ \boldsymbol{\Phi} \end{bmatrix},$$

have full rank $\kappa + 2\ell - 1$, for all ${}^is \in \mathbb{C}$. Furthermore $\boldsymbol{\Phi}$ is a **polynomial matrix** in the variables ${}^is$, while $\mathbf{W}$ and $\mathbf{G}$ are **constant**.

Remarks on the new realization of multivariate rational functions

• Since $\mathbf{\Delta}$ is unimodular, using Schur complements, the dimension of the realization can be reduced to $\kappa+\ell-1$:

$$\hat{\mathbf{\Phi}}=\left[\begin{array}{c|c}\mathbf{\Gamma}(1:\kappa-1,:) & \mathbf{0}_{\kappa-1,\ell-1}\\ \hline \mathbb{A} & \mathbf{\Delta}(1:\ell-1,:)^T\end{array}\right],\ \hat{\mathbf{G}}=\left[\begin{array}{c}\mathbf{0}\\ \hline \mathbf{\Delta}(\ell,:)^T\end{array}\right],\ \hat{\mathbf{W}}=\mathbf{e}_\ell^T\,\mathbf{\Delta}^{-T}\left[\ \mathbb{B}\ \mid\ \mathbf{0}_{\ell,\ell-1}\ \right].$$

This is achieved at the expense of introducing **parameter dependence** in $\hat{\mathbf{W}}$.

• The above expression achieves the **multi-linearization** of the undelying **NEP** (nonlinear eigenvalue problem).

In the case of 2 variables and separation in left and right, we actually achieve a **linearization** of the underlying **NEP**.

• Key technical quantities: the **unimodular matrices** constructed for each variable.

• The possibility of splitting the variables to **left** variables and **right** variables, allows us to pick the splitting that **minimizes** n. For instance, if we have 4 variables with degrees $2,2,1,1$, splitting the variables into $(2,1)$–$(2,1)$ gives $n=17$, while the splitting (2)–$(2,1,1)$ (i.e. one column and 3 rows variables) gives $n=26$.

• It is conjectured that the size of the blue realization is the **smallest** possible with **W** and **G** **constant** (i.e. parameter independent).

Structure of the nullspace of ND-Loewner matrices

Issue: computing the **barycentric weights** $a_{i_1 i_2 \dots i_k}$, requires $\Pi_{i=1}^{N} \nu_i^3$ flops.

Example with: $n = 3$, $m = 2$: $\mathbf{H}(s,t) = \frac{s^3 t}{s - t^2 + 1}$, **interpolation points chosen**:

$$\left\{ \begin{array}{c|c|c|c|c|c|c|c} s_1 = 0 & s_2 = 1 & s_3 = 2 & s_4 = \frac{1}{2} & s_5 = \frac{3}{2} & s_6 = \frac{3}{4} & s_7 = \frac{2}{5} & s_8 = -\frac{2}{5} \\ t_1 = -\frac{1}{4} & t_2 = -\frac{3}{4} & t_3 = -\frac{5}{4} & t_4 = -\frac{1}{2} & t_5 = -1 & t_6 = -\frac{3}{2} & t_7 = \frac{7}{2} & t_8 = \frac{7}{3} \end{array} \right. \Rightarrow$$

$$\mathbf{Tableau} = \left[\begin{array}{c|ccc|ccc||cc} & t_1 & t_2 & t_3 & t_4 & t_5 & t_6 & t_7 & t_8 \\ \hline s_1 & h_{11} & h_{12} & h_{13} & h_{14} & h_{15} & \mathbf{h}_{16} & h_{17} & h_{18} \\ s_2 & h_{21} & h_{22} & h_{23} & h_{24} & h_{25} & \mathbf{h}_{26} & h_{27} & h_{28} \\ s_3 & h_{31} & h_{32} & h_{33} & h_{34} & h_{35} & \mathbf{h}_{36} & h_{37} & h_{38} \\ s_4 & h_{41} & h_{42} & h_{43} & h_{44} & h_{45} & \mathbf{h}_{46} & h_{47} & h_{48} \\ \hline s_5 & \mathbf{h}_{51} & \mathbf{h}_{52} & \mathbf{h}_{53} & \mathbf{h}_{54} & \mathbf{h}_{55} & \mathbf{h}_{56}\ \mathbf{h}_{56} & h_{57} & h_{58} \\ s_6 & \mathbf{h}_{61} & \mathbf{h}_{62} & \mathbf{h}_{63} & \mathbf{h}_{64} & \mathbf{h}_{65} & \mathbf{h}_{66}\ \mathbf{h}_{66} & h_{67} & h_{68} \\ s_7 & \mathbf{h}_{71} & \mathbf{h}_{72} & \mathbf{h}_{73} & \mathbf{h}_{74} & \mathbf{h}_{75} & \mathbf{h}_{76}\ \mathbf{h}_{76} & h_{77} & h_{78} \\ s_8 & \mathbf{h}_{81} & \mathbf{h}_{82} & \mathbf{h}_{83} & \mathbf{h}_{84} & \mathbf{h}_{85} & \mathbf{h}_{86}\ \mathbf{h}_{86} & h_{87} & h_{88} \end{array} \right] =$$

$$= \left[\begin{array}{ccc|ccc||cc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{-4}{31} & \frac{-12}{23} & \frac{-20}{7} & \frac{-2}{7} & -1 & 6 & \frac{-14}{41} & \frac{-21}{31} \\ \frac{-32}{47} & \frac{-32}{13} & \frac{-160}{23} & \frac{-16}{11} & -4 & -16 & \frac{-112}{37} & \frac{-84}{11} \\ \frac{-1}{46} & \frac{-1}{10} & \frac{5}{2} & \frac{-1}{20} & \frac{-1}{4} & \frac{1}{4} & \frac{-7}{172} & \frac{-21}{284} \\ \hline \frac{-9}{26} & \frac{-81}{62} & \frac{-9}{2} & \frac{-3}{4} & \frac{-9}{4} & \frac{-81}{4} & \frac{-63}{52} & \frac{-567}{212} \\ \frac{-1}{16} & \frac{-81}{304} & \frac{-45}{16} & \frac{-9}{64} & \frac{-9}{16} & \frac{81}{64} & \frac{-9}{64} & \frac{-81}{304} \\ \frac{-32}{2675} & \frac{-96}{1675} & \frac{32}{65} & \frac{-16}{575} & \frac{-4}{25} & \frac{48}{425} & \frac{-16}{775} & \frac{-12}{325} \\ \frac{32}{1075} & \frac{32}{25} & \frac{-32}{385} & \frac{16}{175} & \frac{-4}{25} & \frac{-16}{275} & \frac{112}{5825} & \frac{84}{2725} \end{array} \right].$$

$$
\underbrace{\begin{bmatrix}
-2 & -2 & -\frac{54}{5} & -\frac{3}{4} & -1 & \frac{27}{20} & -\frac{32}{115} & -\frac{8}{15} & \frac{96}{425} & -\frac{32}{35} & \frac{8}{15} & \frac{32}{275} \\
2 & -6 & -18 & \frac{3}{4} & -3 & \frac{9}{4} & \frac{32}{115} & -\frac{8}{5} & \frac{32}{85} & \frac{32}{35} & \frac{8}{5} & \frac{32}{165} \\
\frac{2}{3} & 6 & -54 & \frac{1}{4} & 3 & \frac{27}{4} & \frac{32}{345} & \frac{8}{5} & \frac{96}{85} & \frac{32}{105} & -\frac{8}{5} & \frac{32}{55} \\
-\frac{154}{31} & -\frac{526}{93} & -\frac{998}{31} & \frac{23}{124} & \frac{215}{93} & -\frac{2767}{620} & -\frac{7216}{10695} & \frac{32}{465} & -\frac{12752}{39525} & -\frac{4784}{7595} & \frac{32}{1085} & -\frac{2416}{59675} \\
\frac{42}{23} & -\frac{318}{23} & -\frac{1210}{23} & \frac{561}{92} & \frac{15}{23} & -\frac{877}{92} & \frac{1136}{345} & -\frac{832}{345} & -\frac{8272}{5865} & \frac{9872}{5635} & -\frac{832}{805} & -\frac{11728}{26565} \\
-\frac{118}{21} & -\frac{34}{7} & -\frac{974}{7} & \frac{1217}{84} & \frac{257}{7} & -\frac{1847}{28} & \frac{15184}{2415} & \frac{1888}{105} & -\frac{35344}{1785} & \frac{688}{245} & \frac{1888}{245} & -\frac{21552}{2695} \\
\frac{26}{47} & \frac{590}{141} & \frac{7358}{235} & -\frac{325}{188} & -\frac{89}{705} & -\frac{1171}{940} & -\frac{8824}{5405} & -\frac{102}{235} & -\frac{7928}{19975} & -\frac{6352}{4935} & -\frac{68}{235} & -\frac{8048}{38775} \\
\frac{178}{13} & -\frac{22}{13} & \frac{1850}{39} & \frac{1931}{260} & -\frac{79}{13} & -\frac{3101}{780} & \frac{9096}{1495} & -\frac{374}{65} & -\frac{7112}{3315} & \frac{1936}{455} & -\frac{748}{195} & -\frac{2864}{2145} \\
\frac{1142}{69} & \frac{866}{23} & \frac{2446}{23} & \frac{10033}{1380} & \frac{2353}{115} & -\frac{12103}{460} & \frac{664}{115} & \frac{1954}{115} & -\frac{34552}{1955} & \frac{3152}{805} & \frac{3908}{345} & -\frac{14544}{1265} \\
-\frac{67}{23} & -\frac{205}{69} & -\frac{1861}{115} & -\frac{175}{92} & -\frac{199}{69} & \frac{379}{92} & \frac{28}{115} & \frac{212}{115} & -\frac{10532}{9775} & -\frac{3644}{7245} & \frac{212}{1035} & \frac{1844}{56925} \\
\frac{13}{5} & -\frac{43}{5} & -\frac{403}{15} & \frac{13}{20} & -\frac{37}{5} & \frac{437}{60} & \frac{332}{115} & \frac{12}{5} & -\frac{724}{255} & \frac{268}{315} & \frac{4}{15} & -\frac{92}{1485} \\
\frac{13}{3} & 19 & -91 & \frac{169}{12} & 49 & -\frac{79}{4} & -\frac{3876}{115} & -\frac{532}{5} & \frac{8116}{85} & -\frac{1124}{315} & -\frac{532}{45} & \frac{1876}{165}
\end{bmatrix}}_{\mathbb{L}_{3D} \in \mathbb{R}^{12 \times 12}},
$$

$$
\Rightarrow \underbrace{\begin{bmatrix} \frac{184}{121} & -\frac{736}{363} & \frac{184}{1089} & -\frac{1216}{231} & \frac{1216}{231} & \frac{1216}{693} & \frac{10051}{2541} & -\frac{6992}{2541} & -\frac{7429}{2541} & -\frac{7}{33} & -\frac{16}{33} & 1 \end{bmatrix}^T}_{\mathcal{N}(\mathbb{L}_{3D})}.
$$

The following **observation** holds:

$$L_{s_5} = \begin{bmatrix} \frac{21}{13} & \frac{33}{13} & \frac{207}{13} \\ \frac{69}{31} & \frac{117}{31} & \frac{783}{31} \\ 5 & 9 & 63 \end{bmatrix}, \quad L_{s_6} = \begin{bmatrix} \frac{5}{16} & \frac{2}{3} & -\frac{17}{16} \\ \frac{153}{304} & \frac{45}{38} & -\frac{621}{304} \\ \frac{57}{16} & 9 & -\frac{261}{16} \end{bmatrix}, \quad L_{s_7} = \begin{bmatrix} \frac{3904}{61525} & \frac{528}{2675} & -\frac{4544}{45475} \\ \frac{4544}{38525} & \frac{688}{1675} & -\frac{6464}{28475} \\ -\frac{5184}{7475} & -\frac{848}{325} & \frac{8384}{5525} \end{bmatrix},$$

$$L_{s_8} = \begin{bmatrix} -\frac{1856}{7525} & \frac{272}{1075} & \frac{832}{11825} \\ -\frac{832}{175} & \frac{144}{25} & \frac{1472}{825} \\ \frac{64}{275} & -\frac{592}{1925} & -\frac{192}{1925} \end{bmatrix}, \quad L_{t_6} = \begin{bmatrix} -\frac{27}{2} & \frac{27}{16} & \frac{24}{85} & \frac{8}{55} \\ -\frac{105}{2} & \frac{303}{16} & \frac{834}{85} & \frac{238}{55} \\ \frac{17}{2} & -\frac{221}{16} & -\frac{856}{85} & -\frac{1096}{165} \\ -\frac{41}{2} & \frac{65}{16} & \frac{233}{170} & \frac{113}{330} \end{bmatrix} \quad \Rightarrow$$

$$\mathcal{N}(L_{s_5}) = \begin{bmatrix} 9 \\ -12 \\ 1 \end{bmatrix}, \quad \mathcal{N}(L_{s_6}) = \begin{bmatrix} -3 \\ 3 \\ 1 \end{bmatrix}, \quad \mathcal{N}(L_{s_7}) = \begin{bmatrix} -\frac{23}{17} \\ \frac{16}{17} \\ 1 \end{bmatrix}, \quad \mathcal{N}(L_{s_8}) = \begin{bmatrix} -\frac{7}{33} \\ -\frac{16}{33} \\ 1 \end{bmatrix}$$

$$, \quad \mathcal{N}(L_{t_6}) = \begin{bmatrix} \frac{184}{1089} \\ \frac{1216}{693} \\ -\frac{7429}{2541} \\ 1 \end{bmatrix} \quad \Rightarrow \quad \mathcal{N}(\mathbb{L}_{3D}) = \left[\begin{array}{c} \frac{184}{121} \\ -\frac{736}{363} \\ \frac{184}{1089} \\ \hline -\frac{1216}{231} \\ \frac{1216}{231} \\ \frac{1216}{693} \\ \hline \frac{10051}{2541} \\ -\frac{6992}{2541} \\ -\frac{7429}{2541} \\ \hline -\frac{7}{33} \\ -\frac{16}{33} \\ 1 \end{array}\right]$$

$\Rightarrow$ 172 *vs* 1728 *flops*

$\rightarrow$ **decoupling of the variables**

Explanation: Constructing nullvectors of 2D from nullvectors of 1D Loewner matrices.

Consider the case $n = 2$, $m = 2$. The numerator and denominator can be expressed as:

$$\mathbf{num^{2D}} =$$

$$\begin{aligned} & a_{44}h_{44}(s-s_5)(s-s_6)(t-t_5)(t-t_6) + a_{45}h_{45}(s-s_5)(s-s_6)(t-t_4)(t-t_6) + a_{46}h_{46}(s-s_5)(s-s_6)(t-t_4)(t-t_5) \\ & + a_{54}h_{54}(s-s_4)(s-s_6)(t-t_5)(t-t_6) + a_{55}h_{55}(s-s_4)(s-s_6)(t-t_4)(t-t_6) + a_{56}h_{56}(s-s_4)(s-s_6)(t-t_4)(t-t_5) \\ & + a_{64}h_{64}(s-s_4)(s-s_5)(t-t_5)(t-t_6) + a_{65}h_{65}(s-s_4)(s-s_5)(t-t_4)(t-t_6) + a_{66}h_{66}(s-s_4)(s-s_5)(t-t_4)(t-t_5) \end{aligned}$$

$$\frac{a_{44}h_{44}}{(s-s_4)(t-t_4)} + \frac{a_{45}h_{45}}{(s-s_4)(t-t_5)} + \frac{a_{46}h_{46}}{(s-s_4)(t-t_6)} + \frac{a_{54}h_{54}}{(s-s_5)(t-t_4)} + \frac{a_{55}h_{55}}{(s-s_5)(t-t_5)} + \\ + \frac{a_{56}h_{56}}{(s-s_5)(t-t_6)} + \frac{a_{64}h_{64}}{(s-s_6)(t-t_4)} + \frac{a_{65}h_{65}}{(s-s_6)(t-t_5)} + \frac{a_{66}h_{66}}{(s-s_6)(t-t_6)}$$

$$\mathbf{den^{2D}} =$$

$$\begin{aligned} & a_{44}(s-s_5)(s-s_6)(t-t_5)(t-t_6) + a_{45}(s-s_5)(s-s_6)(t-t_4)(t-t_6) + a_{46}(s-s_5)(s-s_6)(t-t_4)(t-t_5) \\ & + a_{54}(s-s_4)(s-s_6)(t-t_5)(t-t_6) + a_{55}(s-s_4)(s-s_6)(t-t_4)(t-t_6) + a_{56}(s-s_4)(s-s_6)(t-t_4)(t-t_5) \\ & + a_{64}(s-s_4)(s-s_5)(t-t_5)(t-t_6) + a_{65}(s-s_4)(s-s_5)(t-t_4)(t-t_6) + a_{66}(s-s_4)(s-s_5)(t-t_4)(t-t_5) \end{aligned}$$

$$\frac{a_{44}}{(s-s_4)(t-t_4)} + \frac{a_{45}}{(s-s_4)(t-t_5)} + \frac{a_{46}}{(s-s_4)(t-t_6)} + \frac{a_{54}}{(s-s_5)(t-t_4)} + \frac{a_{55}}{(s-s_5)(t-t_5)} + \\ + \frac{a_{56}}{(s-s_5)(t-t_6)} + \frac{a_{64}}{(s-s_6)(t-t_4)} + \frac{a_{65}}{(s-s_6)(t-t_5)} + \frac{a_{66}}{(s-s_6)(t-t_6)}$$

The associated **1D** rational functions are obtained by evaluating the 2D numerator/denominator at $s = s_i,\ t = t_j,\ i, j = 1, 2, 3$:

$$\mathbf{num}^{\mathbf{1D}}_{\mathbf{t=t_4}} = \begin{array}{l} a_{44}h_{44}\,(t_4-t_5)\,(t_4-t_6)\,(s-s_5)\,(s-s_6) \\ +a_{54}h_{54}\,(t_4-t_5)\,(t_4-t_6)\,(s-s_4)\,(s-s_6) \\ +a_{64}h_{64}\,(t_4-t_5)\,(t_4-t_6)\,(s-s_4)\,(s-s_5) \end{array}, \quad \mathbf{den}^{\mathbf{1D}}_{\mathbf{t=t_4}} = \begin{array}{l} a_{44}\,(t_4-t_5)\,(t_4-t_6)\,(s-s_5)\,(s-s_6) \\ +a_{54}\,(t_4-t_5)\,(t_4-t_6)\,(s-s_4)\,(s-s_6) \\ +a_{64}\,(t_4-t_5)\,(t_4-t_6)\,(s-s_4)\,(s-s_5) \end{array}$$

$$\mathbf{num}^{\mathbf{1D}}_{\mathbf{t=t_5}} = \begin{array}{l} a_{45}h_{45}\,(t_5-t_4)\,(t_5-t_6)\,(s-s_5)\,(s-s_6) \\ a_{55}h_{55}\,(t_5-t_4)\,(t_5-t_6)\,(s-s_4)\,(s-s_6) \\ a_{65}h_{65}\,(t_5-t_4)\,(t_5-t_6)\,(s-s_4)\,(s-s_5) \end{array}, \quad \mathbf{den}^{\mathbf{1D}}_{\mathbf{t=t_5}} = \begin{array}{l} a_{45}\,(t_5-t_4)\,(t_5-t_6)\,(s-s_5)\,(s-s_6) \\ a_{55}\,(t_5-t_4)\,(t_5-t_6)\,(s-s_4)\,(s-s_6) \\ a_{65}\,(t_5-t_4)\,(t_5-t_6)\,(s-s_4)\,(s-s_5) \end{array}$$

$$\mathbf{num}^{\mathbf{1D}}_{\mathbf{t=t_6}} = \begin{array}{l} a_{46}h_{46}\,(t_4-t_6)\,(t_5-t_6)\,(s-s_5)\,(s-s_6) \\ +a_{56}h_{56}\,(t_4-t_6)\,(t_5-t_6)\,(s-s_4)\,(s-s_6) \\ +a_{66}h_{66}\,(t_4-t_6)\,(t_5-t_6)\,(s-s_4)\,(s-s_5) \end{array}, \quad \mathbf{den}^{\mathbf{1D}}_{\mathbf{t=t_6}} = \begin{array}{l} a_{46}\,(t_4-t_6)\,(t_5-t_6)\,(s-s_5)\,(s-s_6) \\ +a_{56}\,(t_4-t_6)\,(t_5-t_6)\,(s-s_4)\,(s-s_6) \\ +a_{66}\,(t_4-t_6)\,(t_5-t_6)\,(s-s_4)\,(s-s_5) \end{array}$$

$$\mathbf{num}^{\mathbf{1D}}_{\mathbf{s=s_4}} = \begin{array}{l} a_{44}h_{44}\,(s_4-s_5)\,(s_4-s_6)\,(t-t_5)\,(t-t_6) \\ +a_{45}h_{45}\,(s_4-s_5)\,(s_4-s_6)\,(t-t_4)\,(t-t_6) \\ +a_{46}h_{46}\,(s_4-s_5)\,(s_4-s_6)\,(t-t_4)\,(t-t_5) \end{array}, \quad \mathbf{den}^{\mathbf{1D}}_{\mathbf{s=s_4}} = \begin{array}{l} a_{44}\,(s_4-s_5)\,(s_4-s_6)\,(t-t_5)\,(t-t_6) \\ +a_{45}\,(s_4-s_5)\,(s_4-s_6)\,(t-t_4)\,(t-t_6) \\ +a_{46}\,(s_4-s_5)\,(s_4-s_6)\,(t-t_4)\,(t-t_5) \end{array}$$

$$\mathbf{num}^{\mathbf{1D}}_{\mathbf{s=s_5}} = \begin{array}{l} a_{54}h_{54}\,(s_5-s_4)\,(s_5-s_6)\,(t-t_5)\,(t-t_6) \\ a_{55}h_{55}\,(s_5-s_4)\,(s_5-s_6)\,(t-t_4)\,(t-t_6) \\ a_{56}h_{56}\,(s_5-s_4)\,(s_5-s_6)\,(t-t_4)\,(t-t_5) \end{array}, \quad \mathbf{den}^{\mathbf{1D}}_{\mathbf{s=s_5}} = \begin{array}{l} a_{54}\,(s_5-s_4)\,(s_5-s_6)\,(t-t_5)\,(t-t_6) \\ a_{55}\,(s_5-s_4)\,(s_5-s_6)\,(t-t_4)\,(t-t_6) \\ a_{56}\,(s_5-s_4)\,(s_5-s_6)\,(t-t_4)\,(t-t_5) \end{array}$$

$$\mathbf{num}^{\mathbf{1D}}_{\mathbf{s=s_6}} = \begin{array}{l} a_{64}h_{64}\,(s_4-s_6)\,(s_5-s_6)\,(t-t_5)\,(t-t_6) \\ +a_{65}h_{65}\,(s_4-s_6)\,(s_5-s_6)\,(t-t_4)\,(t-t_6) \\ +a_{66}h_{66}\,(s_4-s_6)\,(s_5-s_6)\,(t-t_4)\,(t-t_5) \end{array}, \quad \mathbf{den}_{\mathbf{s=s_6}} = \begin{array}{l} a_{64}\,(s_4-s_6)\,(s_5-s_6)\,(t-t_5)\,(t-t_6) \\ +a_{65}\,(s_4-s_6)\,(s_5-s_6)\,(t-t_4)\,(t-t_6) \\ +a_{66}\,(s_4-s_6)\,(s_5-s_6)\,(t-t_4)\,(t-t_5) \end{array}$$

Example: $\mathbf{H}(s,t,x) = \frac{s^3\,t^3+x^2}{s^4+x\,t^2+3} \Rightarrow [\nu_1\ \nu_2\ \nu_3] = [5\ \ 4\ \ 3] \Rightarrow N = 60.$

Right points: $[\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \mathbf{s}_4, \mathbf{s}_5]$, $[\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4]$, $[\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3]$ $\Rightarrow$

$$\left.\begin{array}{lccccc} \mathbf{S} = & [\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \mathbf{s}_4, \mathbf{s}_5] & \otimes & \mathbb{I}_{1,4} & \otimes & \mathbb{I}_{1,3} \\ \mathbf{T} = & \mathbb{I}_{1,5} & \otimes & [\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4] & \otimes & \mathbb{I}_{1,3} \\ \mathbf{X} = & \mathbb{I}_{1,5} & \otimes & \mathbb{I}_{1,4} & \otimes & [\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3] \end{array}\right\} \in \mathbb{C}^{3\times 60}.$$

5 · 4 · 3				
s_1, t_1, x_1	s_2, t_1, x_1	s_3, t_1, x_1	s_4, t_1, x_1	s_5, t_1, x_1
s_1, t_1, x_2	s_2, t_1, x_2	s_3, t_1, x_2	s_4, t_1, x_2	s_5, t_1, x_2
s_1, t_1, x_3	s_2, t_1, x_3	s_3, t_1, x_3	s_4, t_1, x_3	s_5, t_1, x_3
s_1, t_2, x_1	s_2, t_2, x_1	s_3, t_2, x_1	s_4, t_2, x_1	s_5, t_2, x_1
s_1, t_2, x_2	s_2, t_2, x_2	s_3, t_2, x_2	s_4, t_2, x_2	s_5, t_2, x_2
s_1, t_2, x_3	s_2, t_2, x_3	s_3, t_2, x_3	s_4, t_2, x_3	s_5, t_2, x_3
s_1, t_3, x_1	s_2, t_3, x_1	s_3, t_3, x_1	s_4, t_3, x_1	s_5, t_3, x_1
s_1, t_3, x_2	s_2, t_3, x_2	s_3, t_3, x_2	s_4, t_3, x_2	s_5, t_3, x_2
s_1, t_3, x_3	s_2, t_3, x_3	s_3, t_3, x_3	s_4, t_3, x_3	s_5, t_3, x_3
s_1, t_4, x_1	s_2, t_4, x_1	s_3, t_4, x_1	s_4, t_4, x_1	s_5, t_4, x_1
s_1, t_4, x_2	s_2, t_4, x_2	s_3, t_4, x_2	s_4, t_4, x_2	s_5, t_4, x_2
s_1, t_4, x_3	s_2, t_4, x_3	s_3, t_4, x_3	s_4, t_4, x_3	s_5, t_4, x_3

20 3 × 3
variable x

5 · 4				
s_1, t_1, x_3	s_2, t_1, x_3	s_3, t_1, x_3	s_4, t_1, x_3	s_5, t_1, x_3
s_1, t_2, x_3	s_2, t_2, x_3	s_3, t_2, x_3	s_4, t_2, x_3	s_5, t_2, x_3
s_1, t_3, x_3	s_2, t_3, x_3	s_3, t_3, x_3	s_4, t_3, x_3	s_5, t_3, x_3
s_1, t_4, x_3	s_2, t_4, x_3	s_3, t_4, x_3	s_4, t_4, x_3	s_5, t_4, x_3

5 4 × 4
variable t

5				
s_1, t_4, x_3	s_2, t_4, x_3	s_3, t_4, x_3	s_4, t_4, x_3	s_5, t_4, x_3

1 5 × 5
variable s

$$\underbrace{\begin{array}{ccccc} a_{1,1,1} & a_{2,1,1} & a_{3,1,1} & a_{4,1,1} & a_{5,1,1} \\ a_{1,1,2} & a_{2,1,2} & a_{3,1,2} & a_{4,1,2} & a_{5,1,2} \\ a_{1,1,3} & a_{2,1,3} & a_{3,1,3} & a_{4,1,3} & a_{5,1,3} \\ a_{1,2,1} & a_{2,2,1} & a_{3,2,1} & a_{4,2,1} & a_{5,2,1} \\ a_{1,2,2} & a_{2,2,2} & a_{3,2,2} & a_{4,2,2} & a_{5,2,2} \\ a_{1,2,3} & a_{2,2,3} & a_{3,2,3} & a_{4,2,3} & a_{5,2,3} \\ a_{1,3,1} & a_{2,3,1} & a_{3,3,1} & a_{4,3,1} & a_{5,3,1} \\ a_{1,3,2} & a_{2,3,2} & a_{3,3,2} & a_{4,3,2} & a_{5,3,2} \\ a_{1,3,3} & a_{2,3,3} & a_{3,3,3} & a_{4,3,3} & a_{5,3,3} \\ a_{1,4,1} & a_{2,4,1} & a_{3,4,1} & a_{4,4,1} & a_{5,4,1} \\ a_{1,4,2} & a_{2,4,2} & a_{3,4,2} & a_{4,4,2} & a_{5,4,2} \\ a_{1,4,3} & a_{2,4,3} & a_{3,4,3} & a_{4,4,3} & a_{5,4,3} \end{array}}_{\textbf{Bary}}$$

$$\underbrace{\begin{bmatrix} \frac{331}{385} & -\frac{716}{385} & 1 \\ \hline \frac{857}{1004} & -\frac{1861}{1004} & 1 \\ \hline \frac{541}{637} & -\frac{1178}{637} & 1 \\ \hline \frac{1337}{1580} & -\frac{2917}{1580} & 1 \\ \hline\hline \frac{1339}{1555} & -\frac{2894}{1555} & 1 \\ \hline \frac{247}{289} & -\frac{536}{289} & 1 \\ \hline \frac{2179}{2563} & -\frac{4742}{2563} & 1 \\ \hline \frac{2689}{3175} & -\frac{5864}{3175} & 1 \\ \hline\hline \frac{13}{15} & -\frac{28}{15} & 1 \\ \hline \frac{299}{348} & -\frac{647}{348} & 1 \\ \hline \frac{187}{219} & -\frac{406}{219} & 1 \\ \hline \frac{17}{20} & -\frac{37}{20} & 1 \\ \hline\hline \frac{1579}{1795} & -\frac{3374}{1795} & 1 \\ \hline \frac{1969}{2263} & -\frac{4232}{2263} & 1 \\ \hline \frac{2419}{2803} & -\frac{5222}{2803} & 1 \\ \hline \frac{2929}{3415} & -\frac{6344}{3415} & 1 \\ \hline\hline \frac{487}{541} & -\frac{1028}{541} & 1 \\ \hline \frac{167}{188} & -\frac{355}{188} & 1 \\ \hline \frac{697}{793} & -\frac{1490}{793} & 1 \\ \hline \frac{1649}{1892} & -\frac{3541}{1892} & 1 \end{bmatrix}}_{\mathbf{Bary}_x}, \underbrace{\begin{bmatrix} -\frac{77}{158} \\ \frac{753}{395} \\ -\frac{1911}{790} \\ 1 \\ \hline -\frac{311}{635} \\ \frac{6069}{3175} \\ -\frac{7689}{3175} \\ 1 \\ \hline -\frac{1}{2} \\ \frac{29}{15} \\ -\frac{73}{30} \\ 1 \\ \hline -\frac{359}{683} \\ \frac{6789}{3415} \\ -\frac{8409}{3415} \\ 1 \\ \hline -\frac{541}{946} \\ \frac{987}{473} \\ -\frac{2379}{946} \\ 1 \end{bmatrix}}_{\mathbf{Bary}_t}, \underbrace{\begin{bmatrix} \frac{395}{473} \\ -\frac{3175}{946} \\ \frac{2430}{473} \\ -\frac{3415}{946} \\ 1 \end{bmatrix}}_{\mathbf{Bary}_s} \Rightarrow \boxed{\begin{array}{c} (\mathbf{Bary}_x) \odot (\mathbf{Bary}_t \otimes \mathbb{I}_{3,1}) \odot (\mathbf{Bary}_s \otimes \mathbb{I}_{12,1}) = \mathcal{N}(\mathbb{L}_{3D}) \\ \textbf{\underline{DECOUPLING} of the variables} \end{array}}$$

The ND-case. Consider $\mathbf{H}(\cdots, {}^i s, \cdots)$, of degrees $\nu_i - 1 > 0$, $i = 1, \cdots, N$. The following hold:

# of 1D Loewner matrices $\mathbb{L}$	size of each $\mathbb{L}$	# of flops per $\mathbb{L}$
$\nu_1\ \nu_2\ \cdots\ \nu_{N-2}\ \nu_{N-1}$	ν_N	ν_N^3
$\nu_1\ \nu_2\ \cdots\ \nu_{N-2}$	ν_{N-1}	ν_{N-1}^3
$\vdots$	$\vdots$	$\vdots$
$\nu_1\ \nu_2$	ν_3	ν_3^3
ν_1	ν_2	ν_2^3
1	ν_1	ν_1^3

Remark. The 1D Loewner matrices $\mathbb{L}_{\mathrm{1D}}$, can be computed in parallel (simultanously).

Hence the total **flops/storage** required to compute an element of the null space of $\mathbb{L}_{\mathrm{ND}}$ is:

$$\textbf{Flops}_{\mathrm{1D}} = \nu_1^3 + (\nu_1)\ \nu_2^3 + (\nu_1\nu_2)\ \nu_3^3 + \cdots + (\nu_1\nu_2\cdots\nu_{N-1})\ \nu_N^3 \quad \text{vs.} \quad \textbf{Flops}_{\mathrm{ND}} = \nu_1^3\nu_2^3\cdots\nu_N^3,$$

$$\textbf{Storage}_{\mathrm{1D}} = \nu_1^2 + (\nu_1)\ \nu_2^2 + (\nu_1\nu_2)\ \nu_3^2 + \cdots + (\nu_1\nu_2\cdots\nu_{N-1})\ \nu_N^2 \quad \text{vs.} \quad \textbf{Storage}_{\mathrm{ND}} = (\nu_1\nu_2\cdots\nu_N)^2.$$

Resulting advantages:

(a) massive computational savings, making the procedure feasible.

Instead of computing *ND*-Loewner matrices of size $N = \Pi\nu_i$, one computes 1*D*-Loewner matrices of size ν_i, $i = 1, \cdots, k$.

(b) improved numerical accuracy, again making this procedure feasible in many real situations.

$\Downarrow$

Taming the curse of dimensionality

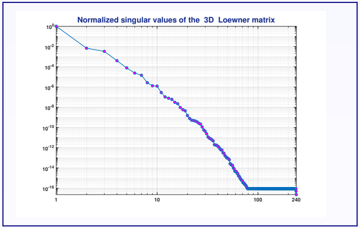
3D example: Numerical issues.

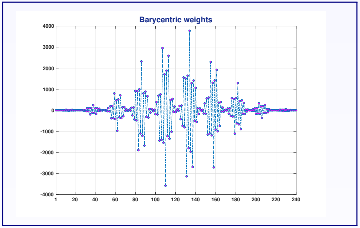
$$\mathbf{H}(s,t,x) = \frac{s^9\, t^7 + s^3 + 5\,x^2}{5\,s^4 + 4\,s^2 + x\,t^3 + 1} \quad \Rightarrow \quad \nu_1 = 10,\ \nu_2 = 8,\ \nu_3 = 3 \quad \Rightarrow \quad N = 240.$$

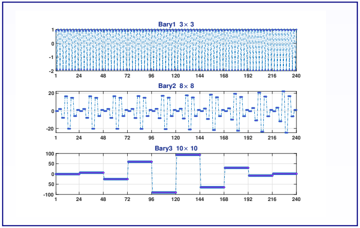
Flops for computing the barycentric weights: $10 \cdot 8 \cdot 3^3 + 10 \cdot 8^3 + 10^3 = \mathbf{8,280}$ vs. $240^3 = \mathbf{13,824,000}$.

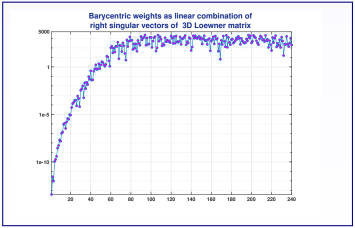
The nullspace of $\mathbb{L}_{3D}$ in floating point arithmetic, has dimension in excess of **100** instead of **1**!

Compute the **barycentric weights** iteratively, by means of **91** 1D Loewner matrices.









Conclusions

- We extended the data-driven model reduction method, based of the **Loewner Framework** to N-parameter systems.
- The approach is closely related to the **barycentric interpolation** in N dimensions.
- Issue: complexity for many parameters, i.e. **C**urse-of-**D**imensionality.
- Solution: reduce the procedure to one involving **1D** Loewner matrices, only.

> This addresses the Curse-of-Dimensionality (**C**-of-**D**) of multi-parameter linear systems by **decoupling** the variables. The computational complexity is thereby reduced by several orders of magnitude.

- In addition to computational effort in many cases this new method improves the **numerical accuracy** of the procedure.
- The data used are **ND-tensors**. Therefore we have provided a way to tame **C**-of-**D** for **ND-tensors**.

Reference:

• A.C. Antoulas, I.-V. Goşea, C. Poussot-Vassal,
The Loewner framework for parametric systems: Taming the Curse-of-Dimensionality,
https://arxiv.org/abs/2405.00495, May 2024.

... and in parting, a 20-D example, courtesy of Charles Poussot-Vassal:

$$\mathbf{H}({}^1s, {}^2s, \cdots, {}^{20}s) =$$

$$\frac{3 \cdot {}^1s^3 + 4 \cdot {}^8s + {}^{12}s + {}^{13}s \cdot {}^{14}s + {}^{15}s - {}^{20}s}{{}^1s + {}^2s^2 \cdot {}^3s + {}^4s + {}^5s + {}^6s + {}^7s \cdot {}^8s + {}^9s \cdot {}^{10}s \cdot {}^{11}s + {}^{13}s + {}^{13}s^3 + {}^{17}s + {}^{18}s \cdot {}^{19}s}$$

Statistics

- Complexity: $(3, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 3, 1, 1, 1, 1)$
- 20-D tensor
- $\mathbb{L}_{\mathrm{20D}}$ has size $6,291,456$.
- $6,291,456^2 \cdot \frac{8}{2^{30}} = 294,912$ GB of storage in double precision
- **Full SVD:** $2.49 \cdot 10^{20}$ `flop`

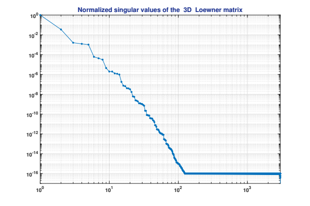
  Cascaded SVD: $5.43 \cdot 10^{7}$ `flop`

Thanks for your attention

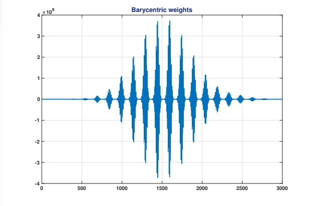
Appendix. A higher-degree example

$\mathbf{H}(s,t,x) = \frac{s^{19}\,t^{12} + s^3 + 5\,x}{5\,s^{19} + 4\,t^{14} + t^3\,x^9 + 1} \quad \rightarrow \quad \nu_1 = 20,\ \nu_2 = 15,\ \nu_3 = 10, \quad N = 3000.$

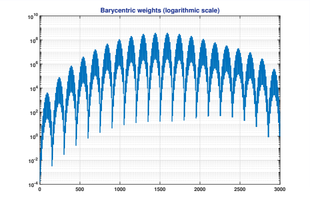
Number of flops required: $300 \cdot 10^3 + 20 \cdot 15^3 + 20^3 = 375,500$ vs $27 \cdot 10^9$ (ratio 1:71,904).



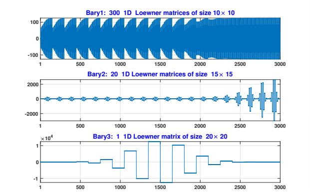
The normalized singular values of $\mathbb{L}_{3D}$



The barycentric weights ($\kappa = 2 \cdot 10^{12}$)



The barycentric weights (logarithmic scale)



Three 1D sequences, whose product yields the 3D weights

Appendix. Data usage: variables s, t, z, degrees $\nu_1 = 3$, $\nu_2 = 2$, $\nu_3 = 2$

$$\left[\begin{array}{c|c|c|c|c|c} \underbrace{\begin{matrix}\mathbf{h}_{111} & \mathbf{h}_{112} \\ \mathbf{h}_{113} & \mathbf{h}_{114}\end{matrix}}_{\mathbb{L}_{1,1,\bullet}} & \underbrace{\begin{matrix}\mathbf{h}_{121} & \mathbf{h}_{122} \\ \mathbf{h}_{123} & \mathbf{h}_{124}\end{matrix}}_{\mathbb{L}_{1,2,\bullet}} & \underbrace{\begin{matrix}\mathbf{h}_{211} & \mathbf{h}_{212} \\ \mathbf{h}_{213} & \mathbf{h}_{214}\end{matrix}}_{\mathbb{L}_{2,1,\bullet}} & \underbrace{\begin{matrix}\mathbf{h}_{221} & \mathbf{h}_{222} \\ \mathbf{h}_{223} & \mathbf{h}_{224}\end{matrix}}_{\mathbb{L}_{2,2,\bullet}} & \underbrace{\begin{matrix}\mathbf{h}_{311} & \mathbf{h}_{312} \\ \mathbf{h}_{313} & \mathbf{h}_{314}\end{matrix}}_{\mathbb{L}_{3,1,\bullet}} & \underbrace{\begin{matrix}\mathbf{h}_{321} & \mathbf{h}_{322} \\ \mathbf{h}_{323} & \mathbf{h}_{324}\end{matrix}}_{\mathbb{L}_{3,2,\bullet}} \end{array}\right] \left\{ \begin{array}{l} \textbf{variable z}: \ 6 \ \mathbb{L}_{1D} \\ \textbf{24 evaluations} \end{array}\right.$$

	z_1				z_2				z_3				z_4			
	t_1	t_2	t_3	t_4	t_1	t_2	t_3	t_4	t_1	t_2	t_3	t_4	t_1	t_2	t_3	t_4
s_1	x	x			x	x										
s_2	x	x			x	x										
s_3	x	x			x	x										
s_4											x	x			x	x
s_5											x	x			x	x
s_6											x	x			x	x

$$\left[\begin{array}{c|c|c} \underbrace{\begin{matrix}\mathbf{h}_{112} & \mathbf{h}_{122} \\ \mathbf{h}_{132} & \mathbf{h}_{142}\end{matrix}}_{\mathbb{L}_{1,\bullet,2}} & \underbrace{\begin{matrix}\mathbf{h}_{212} & \mathbf{h}_{222} \\ \mathbf{h}_{232} & \mathbf{h}_{242}\end{matrix}}_{\mathbb{L}_{2,\bullet,2}} & \underbrace{\begin{matrix}\mathbf{h}_{312} & \mathbf{h}_{322} \\ \mathbf{h}_{332} & \mathbf{h}_{342}\end{matrix}}_{\mathbb{L}_{3,\bullet,2}} \end{array}\right] \left\{ \begin{array}{l} \textbf{variable t}: \ 3 \ \mathbb{L}_{1D} \\ \textbf{12 evaluations (6 new)} \end{array}\right.$$

	z_1				z_2				z_3				z_4			
	t_1	t_2	t_3	t_4	t_1	t_2	t_3	t_4	t_1	t_2	t_3	t_4	t_1	t_2	t_3	t_4
s_1					x	x	x	x								
s_2					x	x	x	x								
s_3					x	x	x	x								
s_4																
s_5																
s_6																

$$\left[\begin{array}{c}\mathbf{h}_{122} \quad \mathbf{h}_{222} \quad \mathbf{h}_{322} \\ \underbrace{\mathbf{h}_{422} \quad \mathbf{h}_{522} \quad \mathbf{h}_{622}}_{\mathbb{L}_{\bullet,2,2}}\end{array}\right] \left\{\begin{array}{l}\textbf{variable s}: \ \mathbf{1} \ \mathbb{L}_{1D} \\ \textbf{6 evaluations (3 new)}\end{array}\right.$$

	z_1					z_2					z_3					z_4			
	t_1	t_2	t_3	t_4		t_1	t_2	t_3	t_4		t_1	t_2	t_3	t_4		t_1	t_2	t_3	t_4
s_1					s_1		x			s_1					s_1				
s_2					s_2		x			s_2					s_2				
s_3					s_3		x			s_3					s_3				
s_4					s_4		x			s_4					s_4				
s_5					s_5		x			s_5					s_5				
s_6					s_6		x			s_6					s_6				

$$\begin{array}{cccc|cccc|cccc|l}
\frac{\mathbf{a}_{111}}{\mathbf{a}_{112}} & 1 & \frac{\mathbf{a}_{121}}{\mathbf{a}_{122}} & 1 & \frac{\mathbf{a}_{211}}{\mathbf{a}_{212}} & 1 & \frac{\mathbf{a}_{221}}{\mathbf{a}_{222}} & 1 & \frac{\mathbf{a}_{311}}{\mathbf{a}_{312}} & 1 & \frac{\mathbf{a}_{321}}{\mathbf{a}_{322}} & 1 & \mathbf{Bary_z} \\
\frac{\mathbf{a}_{112}}{\mathbf{a}_{122}} & \frac{\mathbf{a}_{112}}{\mathbf{a}_{122}} & 1 & 1 & \frac{\mathbf{a}_{212}}{\mathbf{a}_{222}} & \frac{\mathbf{a}_{212}}{\mathbf{a}_{222}} & 1 & 1 & \frac{\mathbf{a}_{312}}{\mathbf{a}_{322}} & \frac{\mathbf{a}_{312}}{\mathbf{a}_{322}} & 1 & 1 & \mathbf{Bary_t} \\
\mathbf{a}_{122} & \mathbf{a}_{122} & \mathbf{a}_{122} & \mathbf{a}_{122} & \mathbf{a}_{222} & \mathbf{a}_{222} & \mathbf{a}_{222} & \mathbf{a}_{222} & \mathbf{a}_{322} & \mathbf{a}_{322} & \mathbf{a}_{322} & \mathbf{a}_{322} & \mathbf{Bary_s} \\
\hline\hline
\mathbf{a}_{111} & \mathbf{a}_{112} & \mathbf{a}_{121} & \mathbf{a}_{122} & \mathbf{a}_{211} & \mathbf{a}_{212} & \mathbf{a}_{221} & \mathbf{a}_{222} & \mathbf{a}_{311} & \mathbf{a}_{312} & \mathbf{a}_{321} & \mathbf{a}_{322} & \mathbf{Bary_{tot}}
\end{array}$$

Total number of function evaluations: 33.

Flops: 99 vs. 1728.