Data-driven feedback control design and stability analysis for complex dynamical systems

SIAM CT21
"Model reduction for control of high-dimensional nonlinear systems"
invited by B. Kramer

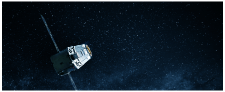
Charles Poussot-Vassal
July, 2021

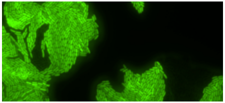
THE FRENCH AEROSPACE LAB

Dynamical models are centrals tools in engineering

Computer-based **advanced modelling** is crucial

- **for verification and validation**
  (μ, $\mathcal{H}_\infty$-norm, pseudo-spectra, Monte Carlo)
- **uncertainty propagation**
  (Multi Disc. Optim., robust optim.)
- **control synthesis**
  ($\mathcal{H}_\infty/\mathcal{H}_2$-norm, MPC, adaptive)





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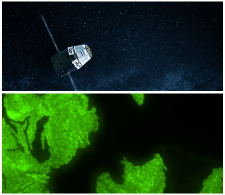

Complex models

- important sim. time
- memory burden
- inaccurate results
- limit model class

Introduction

Complex models...

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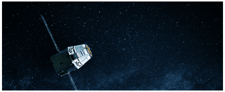

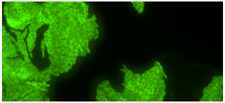

Complex models

- important sim. time
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$\Longrightarrow$

Model simplification

Simplified models

- reduced sim. time
- memory saving
- accurate results
- rational model

Introduction

Connection with control design / stability

Finite models

- spatial meshing of **PDE**

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t)$$

- standard mechanical equations

$$M\ddot{\mathbf{x}}(t) = C\dot{\mathbf{x}}(t) + K\mathbf{x}(t) + B\mathbf{u}(t)$$

- structured....

$$(J - H)\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t)$$

Infinite models / data

- exact solution of linear **PDE**

$$\mathbf{y}(s) = e^{-\sqrt{s}}\mathbf{u}(s)$$

- delays in the loop

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t - \tau) + B\mathbf{u}(t)$$

- measurements on setup

$$\mathbf{y}(z_i) = \mathbf{G}\mathbf{u}(z_i)$$

Control design is well established

Control design is more complex to set

Infinite dimensional dynamical models describe a larger class of systems but remains quite difficult to control

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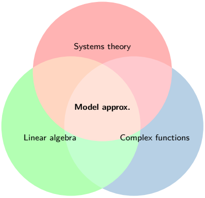
Talk's objectives and messages



Interpolation-based framework is suited for

- for model-based reduction and approximation
- for data-driven model construction

but **may also be** a pivotal tool for

- **data-driven control design** and
- **stability estimation**

of infinite dimensional models and data.

P. Kergus, *"Data-driven control of infinite dimensional systems: Application to a continuous crystallizer"*, IEEE Control Systems Letters.

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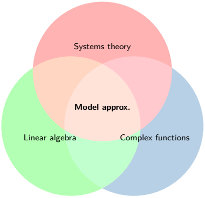

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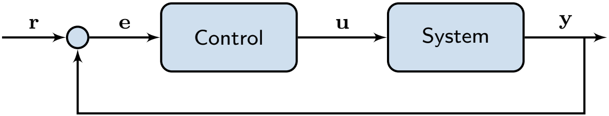

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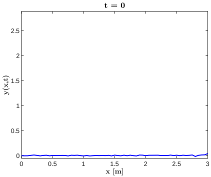
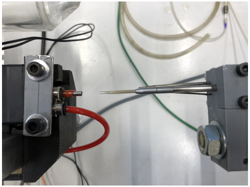
#1 transport phenomena

- A linear **PDE**

$$\begin{array}{rcl} \tilde{\mathbf{y}}_x(\mathbf{x},t) + 2x\tilde{\mathbf{y}}_t(\mathbf{x},t) & = & 0 \\ \tilde{\mathbf{y}}(\mathbf{x},0) & = & 0 \\ \tilde{\mathbf{y}}(0,t) & = & \frac{1}{\sqrt{t}} * \tilde{\mathbf{u}}_f(0,t) \\ \frac{\omega_0^2}{s^2+m\omega_0 s+\omega_0^2}\mathbf{u}(0,s) & = & \mathbf{u}_f(0,s), \end{array}$$

- **L-DDC** vs. Approximation & $\mathcal{H}_\infty$ design



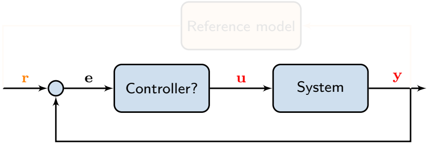


#2 pulsed fluidic actuator

- No model but excitation signals
- Or too complex model
- **L-DDC** design

Data-driven control

VRFT paradigm



L-DDC Data-Driven Control, involves **VRFT** that recasts the control synthesis problem as an identification one

- excite the system with $\mathbf{u}$
- collect $\mathbf{y}$ output signal
- construct the fictive $\mathbf{r}$ reference signal
- construct $\mathbf{e} = \mathbf{r} - \mathbf{y}$
- identify $\mathbf{e} \to \mathbf{u}$ or $\mathbf{u} \to \mathbf{e}$ transfer

$\mathbf{e} \to \mathbf{u}$ is the "Controller" to be identified

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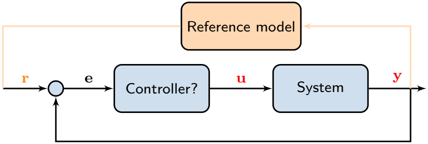

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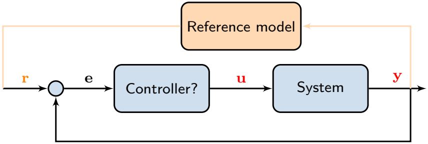

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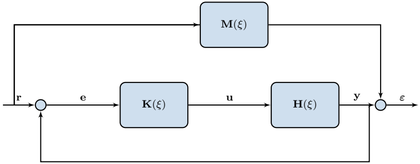

Model-driven VRFT

$$\mathbf{K}^{\star} = \mathbf{H}^{-1}\mathbf{M}(I - \mathbf{M})^{-1}$$

Data-driven VRFT

$$\mathbf{K}^{\star}(z_k) = \mathbf{H}(z_k)^{-1}\mathbf{M}(z_k)\Big(I - \mathbf{M}(z_k)\Big)^{-1}$$

where $\{z_k\}_{k=1}^{N} \in \mathbb{C}$, $k = 1, \ldots, N$

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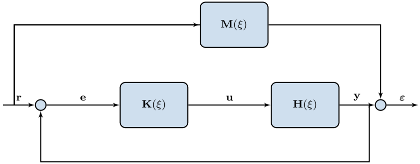

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- **$\mathbf{K}^\star(z_k)$, ideal controller**
- » **Loewner framework**
- » **AAA framework**
- \+ [Formentin/Karimi/...]
- \+ [Ziegler-Nichols/Aström/...]

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- **Interpolation** is flexible,
- adaptable to large set of data
- and with no fixed structure

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Interpolation

Model vs. Data

Model

- L-ODE
- L-ODE / DAE-1
- L-DAE

- L-DDE
- L-PDE

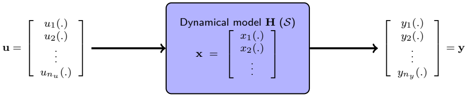
- B-DAE
- Q-DAE

Data

- Data (time)
- Data (frequency)
- Data (parametric)

(Continuous time-domain) $\mathcal{S} \sim \mathbf{u} \to \mathbf{x} \to \mathbf{y}$

(Continuous frequency-domain) $\mathbf{H} \sim \mathbf{u} \to \mathbf{y}$



(Discrete time-domain) $\{t_i, \mathbf{G}(t_i)\}_{i=1}^{N}$

(Discrete frequency-domain) $\{z_i, \mathbf{G}(z_i)\}_{i=1}^{N}$

Interpolation

Deals with model and data

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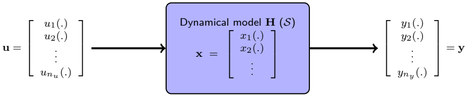

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Interpolation is adapted for many problem, here we focus on **Data-Driven Control** ones.

Rational functions. . . a key ingredient in engineering

Barycentric form (stable and **central in Antoulas, Anderson & Mayo landmark**)

$$\mathbf{H}(z) = \frac{\sum_i \beta_i/(z-\lambda_i)}{\sum_i \alpha_i/(z-\lambda_i)}$$

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Any rational function can be written in the Barycentric form, for any support points λ_i.

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$\Rightarrow$

Rational function simplification

Interpolation

Basis of rational interpolation, model approximation and model reduction tools.

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Given model $\mathbf{H}$, seek $\hat{\mathbf{H}}$ s.t.

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$i = 1, \dots, k$; $j = 1, \dots, q$.

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$\hat{\mathbf{H}}(z) = \mathbf{W}(-z\mathbb{L} + \mathbb{M})^{-1}\mathbf{V} \quad \Rightarrow$ Rational interpolation (data-driven)

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Numerical and experimental examples

Wave equation

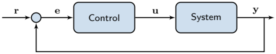
$$
\begin{array}{rcll}
\dfrac{\partial \tilde{y}(x,t)}{\partial x} + 2x\dfrac{\partial \tilde{y}(x,t)}{\partial t} & = & 0 & \text{(transport equation)} \\
\tilde{y}(x,0) & = & 0 & \text{(initial condition)} \\
\tilde{y}(0,t) & = & \dfrac{1}{\sqrt{t}} * \tilde{u}_f(0,t) & \text{(boundary control input)} \\
\dfrac{\omega_0^2}{s^2 + m\omega_0 s + \omega_0^2} u(0,s) & = & u_f(0,s) & \text{(actuator model)}
\end{array}
$$

Equivalent irrational transfer

$$
\begin{array}{rcl}
\mathbf{y}(x,s) & = & \dfrac{\sqrt{\pi}}{\sqrt{s}} e^{-x^2 s} \dfrac{\omega_0^2}{s^2 + m\omega_0 s + \omega_0^2} \mathbf{u}(0,s) \\
& = & \mathbf{G}(x,s)\mathbf{u}(0,s)
\end{array}
$$

Boundary control and measurement

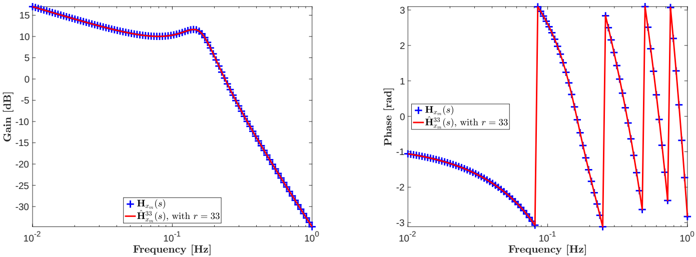
$$
\mathbf{H}_{x_m}(s)\mathbf{u}(s) \leftarrow \mathbf{G}(x_m,s)\mathbf{u}(0,s)
$$



Numerical and experimental examples

Wave equation (a model-based approach)

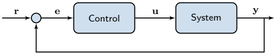
Construct $\hat{\mathbf{H}}$ using Loewner with



Generalised plant

$$
\left\{
\begin{array}{rcl}
\dot{\boldsymbol{\xi}}(t) & = & A_\xi \boldsymbol{\xi}(t) + B_1 \mathbf{r}(t) + B_2 \mathbf{u}(t) \\
\mathbf{z}(t) & = & C_1 \boldsymbol{\xi}(t) + D_{11} \mathbf{r}(t) + D_{12} \mathbf{u}(t) \\
\mathbf{e}(t) & = & \mathbf{r}(t) - \mathbf{y}
\end{array}
\right.
$$

$$
\mathbf{T} = \hat{\mathbf{H}}_{x_m} \mathbf{W}_o
$$



Numerical and experimental examples

Wave equation (a model-based approach)

$\mathcal{H}_\infty$-norm minimisation

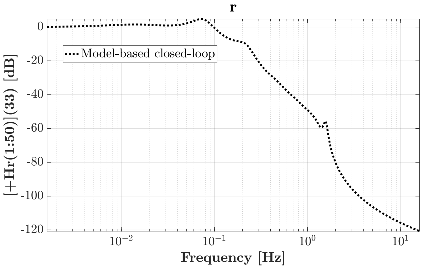
$$\mathbf{K} := \arg\min_{\tilde{\mathbf{K}} \in \mathcal{K}} ||\mathcal{F}_l\left(\mathbf{T}_{\mathbf{rz}}, \tilde{\mathbf{K}}\right)||_{\mathcal{H}_\infty}$$

$$\mathbf{K}(s) = \left(k_p + k_i \frac{1}{s}\right) \frac{1}{s/a+1}$$

`hinfstruct` finds

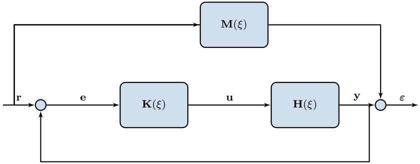
- $k_p = 0.1914$
- $k_i = 0.0251$
- $a = 5667.2$

$$\mathbf{K}(s) = \frac{1084.9(s+0.1313)}{s(s+5667)}$$

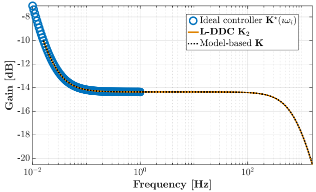


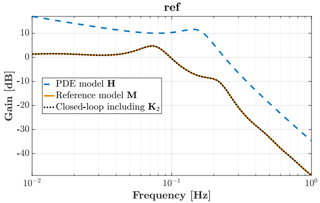
Numerical and experimental examples

Wave equation (data-driven approach)



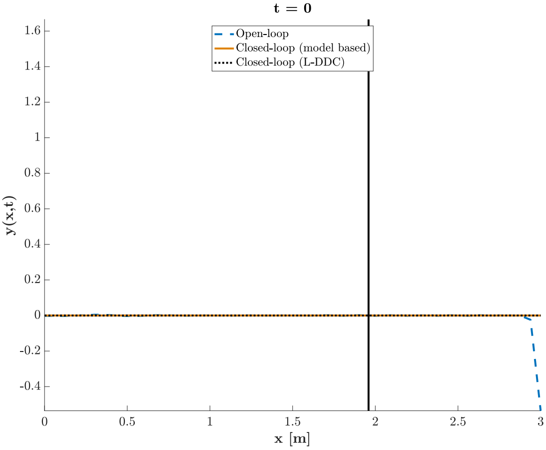
Let $\mathbf{M} \leftarrow \mathbf{T_{rz}}$ and compute **L-DDC**





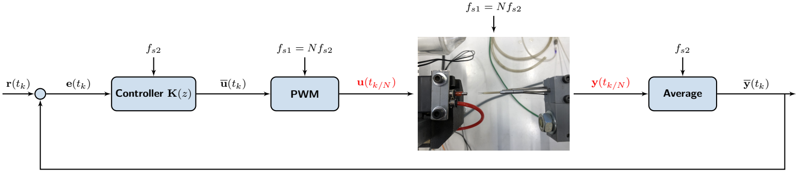
Numerical and experimental examples

Wave equation

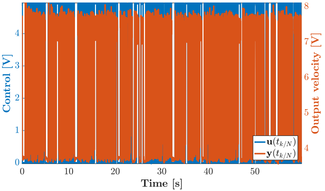


Numerical and experimental examples

Pulsed fluidic actuator



- PFA are ON / OFF actuators,
- blowing air only,
- to modify the pressure,
- measured by a hot wire



C. P-V., P. Kergus, F. Kerhervé, D. Sipp, and L. Cordier, *"Interpolatory-based data-driven pulsed fluidic actuator control design and experimental validation"*, IEEE transactions on Control Systems Technology, 2021.

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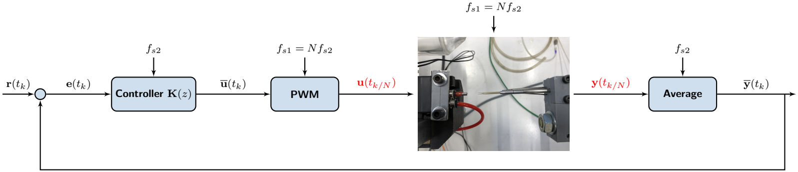

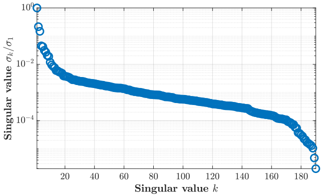

$\mathbf{K}^\star$

and Loewner

- Singular values
- sharp drop,
- $r = 3$ is enough
- $r = 1$ leads to integrator
  Sufficient for stability proof of this positive system

C. Briat, *"A biology-inspired approach to the positive integral control of positive systems: The antithetic, exponential, and logistic integral controllers"*, SIAM J. Appl. Dyn. Syst., 2020..

Numerical and experimental examples

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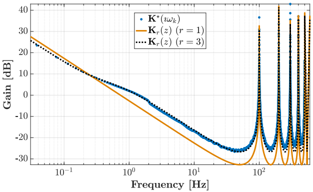

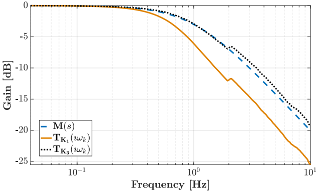

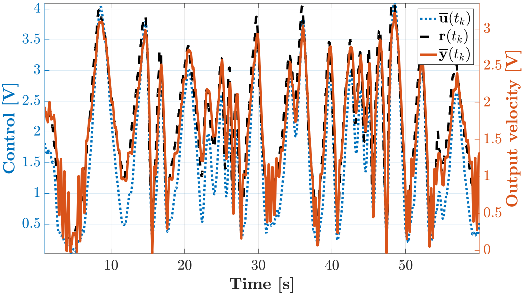

$$\mathbf{H}(s,p) = \frac{1}{s + e^{-ps}}$$

Theoretical stability limit $p = \pi/2$

```
H = @(s) 1./(s+exp(-p*s))
w = logspace(-2,1,100)*2*pi
FR = mor.bode(H,w)
Hr = mor.lti({w,FR},[])
```


M. Kohler, *"On the closest stable descriptor system in the respective spaces $\mathcal{RH}_2$ and $\mathcal{RH}_\infty$"*, Linear Algebra and its Applications, 2014, Vol. 443, pp. 34-49.

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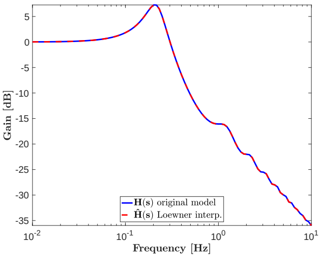
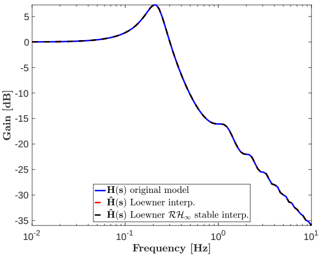
$p = 1$





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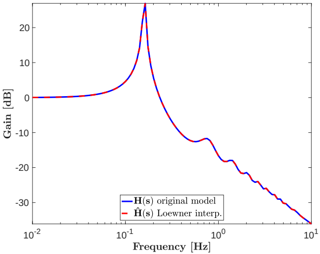

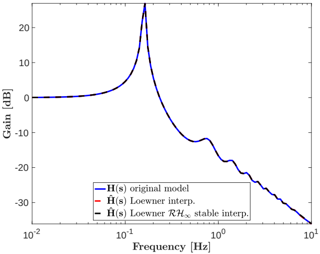

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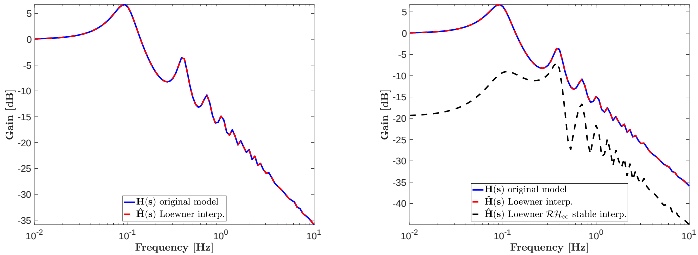
$p = \pi$



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How to address the stability of any $\mathbf{H} \in \mathcal{L}_2$?

Proposed statement

1. Given $\mathbf{H} \in \mathcal{L}_2$, it is possible to find $\hat{\mathbf{H}} \in \mathcal{RL}_2$ that well reproduces $\mathbf{H}$, whatever the complexity of $\mathbf{H}$ is, if we can arbitrarily increase $r = \mathbf{dim}(\hat{\mathbf{H}})$.
   $\Rightarrow$ Can be obtained by increasing the Loewner matrix up to numerical rank loss.

2. If, based on an unstable realisation of $\hat{\mathbf{H}} \in \mathcal{RL}_2$, the optimal stable approximant $\hat{\mathbf{H}}_s \in \mathcal{RH}_\infty$ is close enough to $\hat{\mathbf{H}} \in \mathcal{RL}_2$, in the sense of the $\mathcal{L}_\infty$-norm, then $\hat{\mathbf{H}}$ is stable and, following previous statement (1.), $\mathbf{H}$ is stable too.
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   $\Rightarrow$ ... and a norm computation which threshold is fixed to machine precision.

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$$\mathbf{H} \in \mathcal{L}_2 \xrightarrow{\texttt{Loewner}} \hat{\mathbf{H}} \in \mathcal{RL}_2 \xrightarrow[\texttt{proj.}]{\mathcal{RH}_\infty} \hat{\mathbf{H}}_s \in \mathcal{RH}_\infty \xrightarrow[\texttt{norm}]{\mathcal{L}_\infty} ||\hat{\mathbf{H}}_s - \hat{\mathbf{H}}||_{\mathcal{L}_\infty}$$

$\downarrow$

stable ($< \varepsilon$)
unstable (otherwise)

C. P-V., P. Kergus and P. Vuillemin, *"Interpolation-based irrational model control design and stability analysis"*, Springer, arXiv:2012.01040.

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Stability

$\mathcal{L}_2$ functions stability rationale

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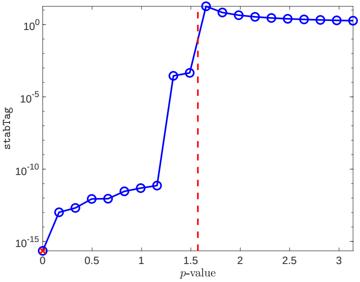
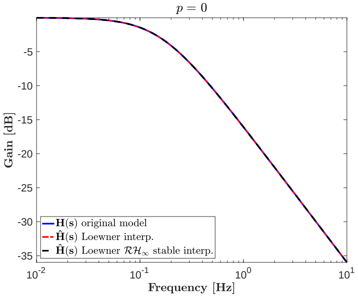
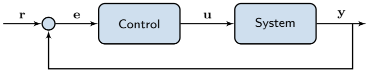




- Model - PI

$$\mathbf{H}(x,s) = \frac{\sqrt{\pi}}{\sqrt{s}} e^{-x^2 s} \frac{\omega_0^2}{s^2 + m\omega_0 s + \omega_0^2}$$

$$\mathbf{K}(0,s) \rightarrow \textbf{PI}$$



Stability

$\mathcal{L}_2$ functions stability rationale (wave equation)

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- Closed-loop

```
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L = @(s) H(s,x)*K(s);
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```

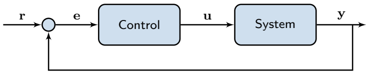

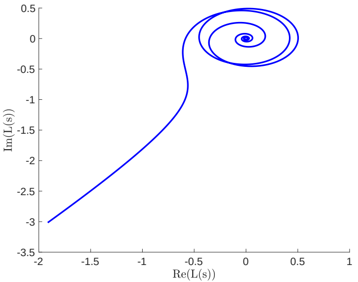

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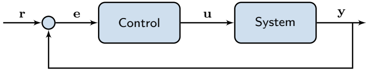

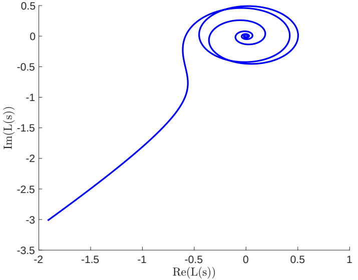

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Conclusions

What to keep in mind...

Interpolation-based methods are remarkably versatile and indicated for

- model reduction and approximation,
- AND control design
- AND complex function stability analysis

$\rightarrow$ **direct impact in simulation engineers**
$\rightarrow$ **in practice complete proof is not ready but may be a staring point...**

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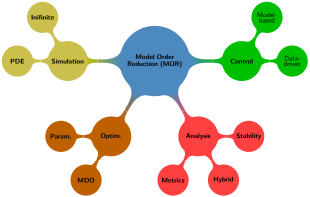
- **Technical references and slides at**
  `sites.google.com/site/charlespoussotvassal/`
- **MOR Toolbox integrated tool at**
  `mordigitalsystems.fr/`



Data-driven feedback control design and stability analysis for complex dynamical systems

SIAM CT21
"Model reduction for control of high-dimensional nonlinear systems"
invited by B. Kramer

Charles Poussot-Vassal
July, 2021

THE FRENCH AEROSPACE LAB