The multivariate Loewner Framework (mLF)

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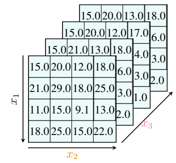
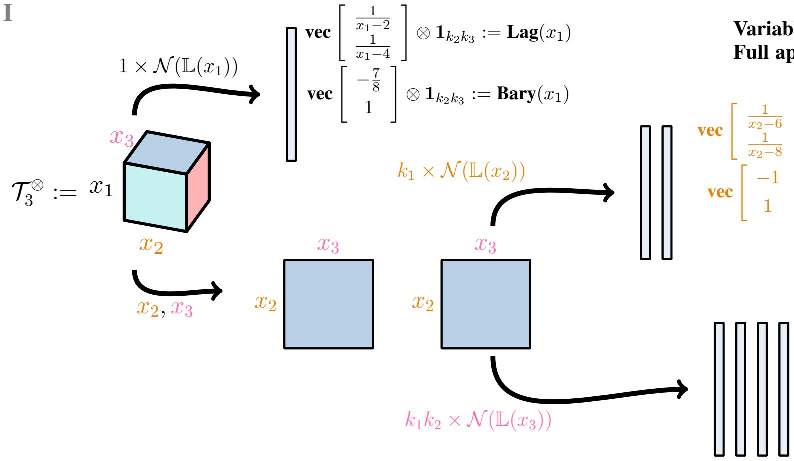
(i) Allows n-variable rational (approximate) functions fitting from any n-D tensor or n-variate function.
(ii) Provides a numerically robust solution to the Kolmogorov Superposition Theorem (KST) for rational functions.
(iii) Bridges "Approximation theory" with "Systems theory", and connects with KAN.
(iv) Allows to treat tensor problems where the data form a full hypercube by means of the matrix framework.

$$\mathbf{H}(x_1, x_2, x_3) = \frac{x_2\,(x_1+1)}{\frac{x_1}{x_3}+1} \Longrightarrow \mathcal{T}_3^{\otimes} \in \mathbb{R}^{4\times4\times4} \text{ along } (x_1, x_2, x_3)$$



I Variable decoupling

Variable decoupling: 56 `flop` and 56 `Bytes`
Full approach: 512 `flop` and 64 `Bytes`



$$\mathbf{vec}\begin{bmatrix}\frac{1}{x_1-2}\\ \frac{1}{x_1-4}\end{bmatrix}\otimes \mathbf{1}_{k_2k_3} := \mathbf{Lag}(x_1)$$

$$\mathbf{vec}\begin{bmatrix}-\frac{7}{8}\\ 1\end{bmatrix}\otimes \mathbf{1}_{k_2k_3} := \mathbf{Bary}(x_1)$$

$$\mathbf{vec}\begin{bmatrix}\frac{1}{x_2-6} & \frac{1}{x_2-6}\\ \frac{1}{x_2-8} & \frac{1}{x_2-8}\end{bmatrix}\otimes \mathbf{1}_{k_2} := \mathbf{Lag}(x_2)$$

$$\mathbf{vec}\begin{bmatrix}-1 & -1\\ 1 & 1\end{bmatrix}\otimes \mathbf{1}_{k_2} := \mathbf{Bary}(x_2)$$

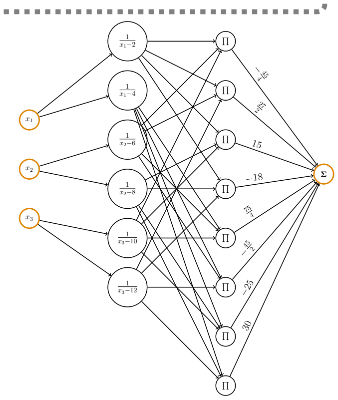
$$\mathbf{vec}\begin{bmatrix}\frac{1}{x_3-10} & \frac{1}{x_3-10} & \frac{1}{x_3-10} & \frac{1}{x_3-10}\\ \frac{1}{x_3-12} & \frac{1}{x_3-12} & \frac{1}{x_3-12} & \frac{1}{x_3-12}\end{bmatrix} := \mathbf{Lag}(x_3)$$

$$\mathbf{vec}\begin{bmatrix}-\frac{6}{7} & -\frac{6}{7} & -\frac{7}{8} & -\frac{7}{8}\\ 1 & 1 & 1 & 1\end{bmatrix} := \mathbf{Bary}(x_3)$$

II KST-like rational form

$$\mathbf{G}_{\text{kst}}(x_1, x_2, x_3) = \frac{\sum_{\text{rows}} \mathbf{w}\odot\mathbf{\Phi}_1(x_1)\odot\mathbf{\Phi}_2(x_2)\odot\mathbf{\Phi}_3(x_3)}{\sum_{\text{rows}} \mathbf{\Phi}_1(x_1)\odot\mathbf{\Phi}_2(x_2)\odot\mathbf{\Phi}_3(x_3)}$$

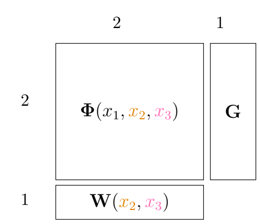
$$\mathbf{\Phi}(x_1) = \begin{bmatrix} -\frac{7}{8\,(x_1-2)}\\ -\frac{7}{8\,(x_1-2)}\\ -\frac{7}{8\,(x_1-2)}\\ -\frac{7}{8\,(x_1-2)}\\ \frac{1}{x_1-4}\\ \frac{1}{x_1-4}\\ \frac{1}{x_1-4}\\ \frac{1}{x_1-4}\end{bmatrix} \quad \mathbf{\Phi}(x_2) = \begin{bmatrix} -\frac{1}{x_2-6}\\ -\frac{1}{x_2-6}\\ \frac{1}{x_2-8}\\ \frac{1}{x_2-8}\\ -\frac{1}{x_2-6}\\ -\frac{1}{x_2-6}\\ \frac{1}{x_2-8}\\ \frac{1}{x_2-8}\end{bmatrix} \quad \mathbf{\Phi}(x_3) = \begin{bmatrix} -\frac{6}{7\,(x_3-10)}\\ \frac{1}{x_3-12}\\ -\frac{6}{7\,(x_3-10)}\\ \frac{1}{x_3-12}\\ -\frac{7}{8\,(x_3-10)}\\ \frac{1}{x_3-12}\\ -\frac{7}{8\,(x_3-10)}\\ \frac{1}{x_3-12}\end{bmatrix} \quad \mathbf{w} = \begin{bmatrix} 15\\ \frac{108}{7}\\ 20\\ \frac{144}{7}\\ \frac{150}{7}\\ \frac{45}{2}\\ \frac{200}{7}\\ 30\end{bmatrix}$$



III Realization

$$\mathbf{G}_{\text{real}}(x_1, x_2, x_3) = \mathbf{W}(x_2, x_3)\mathbf{\Phi}(x_1, x_1, x_2)^{-1}\mathbf{G}$$

$$\mathbf{\Phi}(x_1, x_2, x_3) = \begin{bmatrix} x_1 & -1.0\\ 1.0\,x_3 & 1.0\end{bmatrix} \quad \mathbf{W}(x_2, x_3)^{\top} = \begin{bmatrix} 1.0\,x_2\,x_3\\ 1.0\,x_2\,x_3\end{bmatrix} \quad \mathbf{G} = \begin{bmatrix}0\\1\end{bmatrix}$$



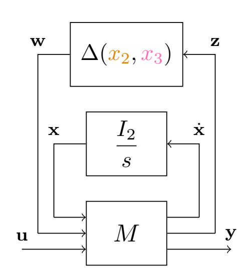
IV LFR uncertain form

$$\mathbf{G}_{\text{lfr}}(x_1, x_2, x_3) = M_{21}\mathbf{\Delta}(E - M_{11}\mathbf{\Delta})^{-1}M_{12}) + M_{22}$$

$$M = \begin{bmatrix} M_{11} & M_{12}\\ M_{21} & M_{22}\end{bmatrix} \quad M_{11} = \left[\begin{array}{cc|ccc} 0 & 1 & 0 & 0 & 0\\ 0 & -1 & 0 & -1 & 0\\ \hline 0 & 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0 & 0\\ 1 & 1 & 0 & 0 & 0\end{array}\right] \quad M_{12} = \begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix} \quad M_{21}^{\top} = \begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix} \quad M_{22} = 0$$

$$E = \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ \hline 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\end{array}\right] \quad \mathbf{\Delta} = \left[\begin{array}{c|c} \frac{I_2}{x_1} & 0\\ \hline 0 & \Delta(x_2, x_3)\end{array}\right] = \left[\begin{array}{cc|ccc} \frac{1}{x_1} & 0 & 0 & 0 & 0\\ 0 & \frac{1}{x_1} & 0 & 0 & 0\\ \hline 0 & 0 & x_2 & 0 & 0\\ 0 & 0 & 0 & x_3 & 0\\ 0 & 0 & 0 & 0 & x_3\end{array}\right]$$

Differential operator $x_1 := s$ and parameters are x_2, x_3.



$s := x_1$ (Laplace variable)

>> A. C. Antoulas, I. V. Gosea and C. Poussot-Vassal, "**On the Loewner framework, the Kolmogorov superposition theorem, and the curse of dimensionality**", in SIAM Review (Research Spotlight), Vol. 67(4), pp. 737-770, November 2025.
>> C. Poussot-Vassal, I. V. Gosea, P. Vuillemin and A. C. Antoulas, "**Tensor-based multivariate function approximation: methods benchmarking and comparison**", `https://arxiv.org/abs/2506.04791`.