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Low-Order Parametric State-Space Modeling of MIMO Systems in the Loewner Framework*

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Abstract. In this work, we present a novel data-driven method for identifying parametric MIMO generalized state-space or descriptor systems of low order that accurately capture the frequency and time domain behavior of large-scale linear dynamical systems. The low-order parametric descriptor systems are identified from transfer matrix samples by means of two-variable Lagrange rational matrix interpolation. This is done within the Loewner framework by deploying the new matrix-valued barycentric formula given in both right and left polynomial matrix fraction forms, which enables the construction of minimal parametric descriptor systems with rectangular transfer matrices. The developed method allows the reduction of order and parameter dependence complexity of the constructed system. Stability of the system is preserved by the postprocessing technique based on flipping signs of unstable poles. The developed methodology is illustrated with a few academic examples and applied to low-order parametric state-space identification of an aerodynamic system.

Key words. system identification, model-order reduction, rational interpolation, Loewner matrix, parametric state-space realization

MSC codes. 34A09, 35B30, 30E10, 41A05, 41A20, 93A15, 93B15, 93B30, 93C35, 93D20

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1. Introduction.

Many engineering applications require repeated simulation and evaluation of the dynamical systems which describe physical phenomena of interest. Therefore, these systems should be computationally efficient and require small storage space. Commonly used models in control engineering are the linear time-invariant (LTI) state-space systems given in the time domain. For many control design purposes, these systems should be of low order, parametric (that is, cover the range of parameters on which the associated physical phenomena depend), and accurate. Additionally, in order to observe and control their dynamic behavior, the state-space systems need to be completely observable and controllable i.e., minimal.

However, physical phenomena are often represented with large-scale linear dynamical systems which form the state-space systems of high orders. An example of this is the semidiscretized partial differential equations (PDEs) over fine meshes for fluid flows. Furthermore, some linear representations of the physical phenomena cannot be brought to the state-space form. Such an example is a representation of an aerodynamic transfer function with a

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nonrational function in the frequency domain. Therefore, for engineering purposes, such as optimal control design, there is a need for constructing parametric state-space systems of low orders which accurately capture the behavior of the original models. Optimal control laws can be used for various engineering purposes, such as load alleviation and flutter suppression in aeronautics. When the state-space description of the large-scale linear system is known, construction of low-order parametric state-space systems is referred to as the *parametric model-order reduction.* If this is not the case, it is referred to as the *low-order parametric system identification.*

Various methods based on rational function approximation/interpolation have been developed with the purpose of constructing the state-space systems from transfer function/matrix samples of the original linear model. An example of a rational approximation method is the vector fitting algorithm, while the famous method based on rational interpolation is the Loewner framework. Vector fitting [14] is an iterative algorithm which constructs a rational approximant by fitting sampled data in the least-squares sense, minimizing the l_2 norm between the samples and the approximant. The algorithm is suitable for both single-input-single-output (SISO) and multiple-input-multiple-output (MIMO) systems. However, good estimation of the starting poles is needed to achieve good numerical conditioning of the least-squares problem. Furthermore, the absence of optimization with respect to the number of poles, which is predefined by the user, can lead to rational approximants that are nonoptimal in terms of accuracy and order. The vector fitting algorithm has been generalized to handle parameter-dependent data in [36, 35, 34, 9, 13] by letting residuals and poles of a rational approximant be parameter-dependent functions. These approaches are, however, inefficient when dealing with a higher number of parameters. For this purpose, Zanco and Grivet-Talocia [41, 42, 44] proposed a novel parameterized framework which adopts an approach based on radial basis functions (RBFs) to capture parametric dependency of the model.

The aforementioned limitations of the vector fitting algorithm can be overcome by using the Loewner framework [2, 1, 25]. With the Loewner framework, generalized state-space or descriptor systems are identified from transfer function/matrix measurements by means of Lagrange rational interpolation. In the nonparametric formulation, this is done by utilizing the barycentric formula or by constructing the Loewner and shifted Loewner matrices directly from the sampled data. Both the approaches allow construction of the minimal SISO systems. However, to guarantee the minimality of MIMO systems, tangential rational interpolation [25] needs to be deployed. So far, this has been achieved only within the latter approach, which utilizes the shifted Loewner matrix. In comparison to the vector fitting algorithm, poles of the rational interpolant follow directly from the sampled data, thus do not need to be initialized, and the framework allows order reduction of the identified systems with the introduction of small approximation errors. Various generalizations of the Loewner framework to parametric formulation have been developed. Two different approaches based on the interpolation of nonparametric models have been proposed by Yue, Feng, and Benner [40] and Kabir and Khazaka [19]. The approach in [40] uses the tangential interpolation from [25] and thus is capable of constructing minimal MIMO realizations, while the approach from [19] deploys the tangential interpolation in matrix form [39, 18] which does not guarantee the minimality of realizations. The main drawback of these approaches is that the identification of the nonparametric descriptor systems at each sampled parameter value is required. A single parametric model, which

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does not suffer from this limitation, is achieved by means of two-variable Lagrange rational function interpolation in the method proposed by Antoulas, Ionita, and Lefteriu [4]. This method utilizes the parameterized barycentric formula and the two-variable Loewner matrix to find the two-variable rational function which interpolates (or approximates) the sampled transfer function data. In this SISO formulation, the construction of minimal realizations is achievable. The method is applied to parametric model-order reduction and generalized to multiple-parameter case by Ionita and Antoulas [17]. Since it is not understood how to apply tangential interpolation using the barycentric formula, generalizing this methodology to achieve a single MIMO parametric model of minimal order is an open question. As a solution to this problem, Lefteriu, Antoulas, and Ionita [23] suggested an approach based on the full interpolation/approximation of transfer matrix samples. This approach, however, is formulated for square transfer matrices, and the procedure that suggests achieving minimal, or close to minimal, state-space models does not hold for arbitrary rectangular transfer matrices. Generalization of this method to rectangular transfer matrices is the topic of this work. An algorithm that combines the Loewner framework with the vector fitting approach was proposed by Nakatsukasa, Sète, and Trefethen [28] under the name Adaptive Anderson–Antoulas (AAA) algorithm. The AAA is an iterative algorithm for rational approximation which utilizes the barycentric formula. Barycentric coefficients are iteratively found such that the interpolation is achieved in chosen support points, while the rest of the data is approximated in the least-squares sense. Various extensions of the algorithm to multi-input and MIMO systems are suggested [15, 5, 10, 12, 24]. In particular, block-AAA introduced in [12] uses the matrix-valued barycentric formula given in the left polynomial matrix fraction form. Surrogate AAA [10], on the other hand, uses the barycentric formula with matrix-valued numerator (while the scalar denominator is common for all the entries), and the barycentric coefficients are found by applying the standard AAA algorithm to a scalar surrogate function. Generalization of the surrogate AAA algorithm to the parametric framework is presented in [31]. However, these approaches only address the problem of finding the rational matrix which approximates the sampled set in the least-squares sense but not its minimal state-space realization.

In addition to matching sampled transfer function/matrix data in the frequency or complex (Laplace) domain, low-order state-space systems need to accurately capture the behavior of the underlying models in the time domain as well. This implies preservation of dynamic stability. This is easily achieved within the nonparametric vector fitting algorithm, while for the parametric framework it is a challenging task. Some solutions to this problem have been proposed [43, 42, 7]. Unlike the nonparametric vector fitting algorithm, the Loewner framework does not offer stability enforcement. Therefore, a suitable postprocessing technique is required to model stable dynamical systems. Commonly used stability enforcement techniques are the stable approximations in the RH_2 and RH_∞ spaces [21] and the sign flipping of unstable poles. These techniques are applied to the systems identified by the Loewner framework and compared in [11]. Accuracy of the stable systems obtained with the approximations in RH_2 and RH_∞ spaces is limited since the found systems are optimal with respect to the identified unstable system, not the original samples. The sign-pole-flipping technique is also limited in its accuracy. Therefore, Carrera-Retana et al. [8] proposed an improved version of the sign-pole-flipping technique. This method involves iterative updating of the system's matrices after the sign flipping of the unstable poles and thus improves the accuracy of the stable systems.

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In this work, we propose a novel method for the identification of parametric MIMO descriptor systems of low-order from transfer matrix samples, based on the two-variable Lagrange rational matrix interpolation. This method generalizes the results from [23] to allow the construction of minimal (or close to minimal) state-space models for nonsquare transfer matrices. To this end, we introduce a new, parameterized matrix-valued barycentric formula in the right and left polynomial matrix fraction forms. This leads to left and right forms of the MIMO two-variable Loewner matrix. Just like its SISO counterpart [17], this method allows construction of a single parametric model rather than multiple nonparametric models which are then interpolated as in [40, 19, 38]. Furthermore, this formulation offers the possibility of choosing separate degrees of the two-variable rational matrix which interpolates/approximates the sampled data in both the complex variable and the parameter variable. Choosing the appropriate form (right or left) of the barycentric formula, based on the shape of the transfer matrix, and the appropriate low degrees of the rational matrix, enables the construction of the low-order state-space systems which can be observed and controlled. The constructed rational interpolant/approximant does not share a common scalar denominator. Thus, if the proposed realization approach is used, this barycentric formula gives smaller realization orders than the one of [31] for the same degree of rational matrix. A unique feature of the approach, which is not available in the vector fitting algorithm, is that the appropriate low degrees of the rational matrix that closely approximates the data are suggested by ranks of the one-variable Loewner matrices, as noted in [17]. Finding such degrees reduces both the order of the system (dictated by the degree in the complex variable) and complexity of its parameter dependence (degree in the parameter variable). We also derive an expression for the pointwise approximation error matrix. To accurately model the system's behavior in the time domain, the developed framework is combined with the improved stabilization technique based on sign flipping of unstable poles [8]. Being data-driven, the proposed methodology can be used for both parametric model-order reduction and low-order system identification. However, it is important to mention that, compared to [40, 31], the method requires higher computational effort which can be prohibitive for large sampled sets and systems with a large number of inputs and outputs. We demonstrate the application of our method to the low-order parametric state-space identification of unsteady aerodynamic loads sampled in the frequency domain. Even though presented for a single parameter, the framework can be extended to multiple parameters in the same fashion as done in [17] for the SISO case. However, with the increasing number of parameters, the associated computational cost and storage become significant challenges. Furthermore, the presented method can be used for other problems that can be expressed in the same form as the complex domain formulation of LTI systems. An example of this, as shown in [27, 26], includes the models resulting from discretization of stationary parametric PDEs.

This work is structured as follows. In section 2, the theoretical background is given and the research problem stated. Next, an overview of the Loewner framework for both nonparametric and parametric SISO systems is given in section 3. In section 4, the developed methodology for MIMO systems is presented, while section 5 covers the stability-preserving postprocessing technique. Finally, in section 6 the developed framework is presented on a few illustrative examples and applied to low-order parametric state-space identification of an aerodynamic system.

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2. Theoretical aspects and problem statement. Parametric LTI descriptor (generalized state-space) system $S(p)$ is given with the set of differential-algebraic equations (DAEs),

$$S(p): \begin{aligned} \mathbf{E}(p)\dot{\mathbf{x}}(t) &= \mathbf{A}(p)\mathbf{x}(t) + \mathbf{B}(p)\mathbf{u}(t), \\ \mathbf{y}(t) &= \mathbf{C}(p)\mathbf{x}(t) + \mathbf{D}(p)\mathbf{u}(t), \end{aligned} \tag{2.1}$$

where $\mathbf{x}(t) \in \mathbb{R}^k$ denotes the internal variable of dimension k, $\mathbf{u}(t) \in \mathbb{R}^{n_u}$ and $\mathbf{y}(t) \in \mathbb{R}^{n_y}$ are the input and output vectors of dimension n_u and n_y, respectively, and $\mathbf{A}(p), \mathbf{E}(p) \in \mathbb{R}^{k\times k}$, $\mathbf{B}(p) \in \mathbb{R}^{k\times n_u}$, $\mathbf{C}(p) \in \mathbb{R}^{n_y\times k}$, $\mathbf{D}(p) \in \mathbb{R}^{n_y\times n_u}$ are the time-invariant, parameter-dependent matrices. Matrix $\mathbf{E}(p)$ is allowed to be singular. The *order* of the system $S(p)$ equals the dimension of the internal variable k. The associated transfer matrix of the system is given as

$$\mathbf{H}(s,p) = \mathbf{C}(p)(s\mathbf{E}(p) - \mathbf{A}(p))^{-1}\mathbf{B}(p) + \mathbf{D}(p). \tag{2.2}$$

We refer to (2.2) as the transfer matrix if the system has multiple inputs or/and multiple outputs ($n_u > 1$ or/and $n_y > 1$), while for SISO systems it is referred to as the transfer function and denoted with $H(s,p)$. Quintuple of matrices $(\mathbf{E},\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D})(p)$ is called the *generalized or descriptor realization* of $\mathbf{H}(s,p)$. We will not distinguish between the descriptor system and the realization of its transfer matrix/function. In general, transfer matrix $\mathbf{H}(s,p)$ is a rational matrix in the complex variable s with parameter-dependent polynomial coefficients. In this work, we seek a descriptor system with a transfer matrix that is rational in both s and p, i.e., two-variable rational matrix. Complexity of a rational matrix is measured by *(McMillan) degree.* A two-variable rational matrix has degrees in both s and p, denoted as n and m. Degree n/m in the complex variable s/parameter variable p is defined as the total number of poles $\rho(p)/\gamma(s)$ of rational matrix when the parameter variable p/complex variable s is held fixed. Properness of $\mathbf{H}(s,p)$ is also defined in both s and p. For a fixed value of p, $\mathbf{H}(s,p)$ can be *strictly proper* ($\lim_{s\to\infty}\mathbf{H}(s,p) = \mathbf{0}$), *proper* ($\lim_{s\to\infty}\mathbf{H}(s,p) = const.$) and *improper* ($\lim_{s\to\infty}\mathbf{H}(s,p) = \infty$) in the variable s. Properness in the parameter variable p is defined analogously. The aforementioned properties hold for a two-variable rational function $H(s,p)$ as well.

In this work, we distinguish between two types of controllability and observability of descriptor systems, namely, R-controllability/observability and C-controllability/observability.

Definition 2.1. *Descriptor system* (2.1) *and the triplet* $(\mathbf{E},\mathbf{A},\mathbf{B})(p)$ *are called controllable on the reachable set or R-controllable if*

$$\mathit{rank}\left[s\mathbf{E}(p) - \mathbf{A}(p), \mathbf{B}(p)\right] = k \text{ for all finite } s \in \mathbb{C}, p \in \mathbb{C}. \tag{2.3}$$

Descriptor system (2.1) *and the triplet* $(\mathbf{E},\mathbf{A},\mathbf{B})(p)$ *are called completely controllable or C-controllable if* (2.3) *holds and*

$$\mathit{rank}\left[\mathbf{E}(p), \mathbf{B}(p)\right] = k \text{ for all } p \in \mathbb{C}. \tag{2.4}$$

Definition 2.2. *Descriptor system* (2.1) *and the triplet* $(\mathbf{E},\mathbf{A},\mathbf{C})(p)$ *are called observable on the reachable set or R-observable if*

$$\mathit{rank}\begin{bmatrix} s\mathbf{E}(p) - \mathbf{A}(p) \\ \mathbf{C}(p) \end{bmatrix} = k \text{ for all finite } s \in \mathbb{C}, p \in \mathbb{C}. \tag{2.5}$$

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Descriptor system (2.1) *and the triplet* $(\mathbf{E}, \mathbf{A}, \mathbf{C})(p)$ *are called completely observable or C-observable if* (2.5) *holds and*

$$\text{rank} \begin{bmatrix} \mathbf{E}(p) \\ \mathbf{C}(p) \end{bmatrix} = k \text{ for all } p \in \mathbb{C}. \tag{2.6}$$

Realization $(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})(p)$ of $\mathbf{H}(s,p)$ (2.2) is *minimal* if and only if the system (2.1) is both C-controllable and C-observable. Definitions 2.1 and 2.2 are parametric versions of the definitions given in [33, 25]. In [25], alternative conditions for C-controllability and C-observability of the descriptor system are also provided.

For the minimal realizations, the poles $\rho(p)$ of $\mathbf{H}(s,p)$ are equal to the generalized eigenvalues of matrix pencil $(\mathbf{A}(p), \mathbf{E}(p))$ denoted as $\Lambda(\mathbf{A}(p), \mathbf{E}(p))$. Descriptor system $S(p)$ is asymptotically stable if $\Lambda(\mathbf{A}(p), \mathbf{E}(p))$ are restricted to the left half of the complex plane, $\mathcal{R}(\Lambda(\mathbf{A}(p), \mathbf{E}(p))) < 0$.

Problem statement. Given the complex domain data

$$\{s_i, p_j, \mathbf{\Phi}_{ij} \mid s_i \in \mathbb{C}, p_j \in \mathbb{C}, \mathbf{\Phi}_{ij} \in \mathbb{C}^{n_y \times n_u}\}, i = 1:N, j = 1:M, \tag{2.7}$$

obtained by sampling the transfer matrix of a parameter-dependent linear system, we seek a parametric descriptor system $\hat{S}(p)$ of low order $\hat{k}$ such that its transfer matrix $\hat{\mathbf{H}}(s,p)$ of degree $(\hat{n}, \hat{m})$ closely approximates the sampled data, $\hat{\mathbf{H}}(s_i, p_j) \approx \mathbf{\Phi}_{ij}$ for $i = 1:N, j = 1:M$. In this work, a single parameter p is considered.

The developed methodology seeks the solution of the stated problem by means of two-variable Lagrange rational matrix interpolation, such that the following conditions hold.

- The identified model $\hat{S}(p)$ is of low order $\hat{k}$ and has low complexity of the parameter dependence $\hat{m}$.
- The identified model is C-controllable and C-observable for rectangular transfer matrix samples.
- The model closely approximates the sampled data given with (2.7) such that the pointwise error matrices, $\mathbf{E}_{rr}(s_i, p_j) = \hat{\mathbf{H}}(s_i, p_j) - \mathbf{\Phi}_{ij}$, have small norms.
- The model accurately captures the dynamic behavior of the original system in the time domain.

When the methodology is used for model-order reduction, the samples $\mathbf{\Phi}_{ij}$ are generated by sampling a transfer function/matrix of a parameter-dependent large-scale dynamical system $S(p)$ described with the state-space equations of high order k. This system is then approximated with the system $\hat{S}(p)$ of lower order $\hat{k}$, $\hat{k} \ll k$. In the case of the system identification, low-order parametric descriptor system $\hat{S}(p)$ is identified from general transfer function/matrix samples of an unknown system. Here, generality implies that the samples do not have to originate from a rational function/matrix.

3. Overview of the Loewner framework for SISO systems.

The problem stated in section 2 is solved in the Loewner framework by means of Lagrange rational interpolation. For the nonparametric systems, this can be done either by introducing the shifted Loewner matrix [25] or by utilizing the barycentric formula [2], in addition to the Loewner matrix, as explained in [16]. The latter approach can be generalized to solve the two-variable rational interpolation

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problem in a straightforward manner. Therefore, we discuss the nonparametric [2] (subsection 3.1) and parametric Loewner frameworks for SISO systems [4, 17] (subsection 3.2), which utilize the barycentric formula, prior to generalizing them to MIMO systems.

3.1. Nonparametric SISO case.

We summarize the results for the one-variable rational function interpolation problem within the Loewner framework [2, 16] and its application to identification of low-order descriptor systems.

3.1.1. One-variable rational function interpolation.

In the one-variable rational function interpolation problem, a set of transfer function samples Φ_i $(n_u = n_y = 1)$, solely dependent on the complex variable s,

$$\{s_i, \Phi_i \mid s_i \in \mathbb{C}, \Phi_i \in \mathbb{C}\}, \quad i = 1:N, \tag{3.1}$$

is given. Here, it is assumed that the sample points s_i are distinct. A unique rational function $H(s)$ which interpolates the given set (3.1), $H(s_i) = \Phi_i, i = 1:N$, and its minimal descriptor realization $(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$, are found by means of Lagrange rational function interpolation. Rational function of degree n is defined as a ratio of two polynomials, $n(s)$ and $d(s)$,

$$H(s) = \frac{n(s)}{d(s)} = \frac{\sum_{i=1}^{n+1} \beta_i l_{s_i}(s)}{\sum_{i=1}^{n+1} \alpha_i l_{s_i}(s)}, \quad \alpha_i \neq 0, \tag{3.2}$$

expressed in Lagrange basis, $s - \lambda_i$. Here $l_{s_i}(s) = \prod_{i'=1, i' \neq i}^{n+1} (s - \lambda_{i'})$ are the Lagrange factors, and $\lambda_{i'}$ denote distinct Lagrange nodes [16]. β_i and α_i are the numerator and denominator coefficients. Equation (3.2) can also be expressed in the rational barycentric form [2, 6],

$$H(s) = \frac{\sum_{i=1}^{n+1} \frac{\beta_i}{s - \lambda_i}}{\sum_{i=1}^{n+1} \frac{\alpha_i}{s - \lambda_i}}. \tag{3.3}$$

The first step in constructing the rational function $H(s)$ is to partition the data (3.1) into two disjoint sets,

$$[s_1, \ldots, s_N] = [\lambda_1, \ldots, \lambda_{\overline{n}}] \cup [\mu_1, \ldots, \mu_{\underline{n}}],$$
$$[\Phi_1, \ldots, \Phi_N] = [w_1, \ldots, w_{\overline{n}}] \cup [\mu_1, \ldots, \mu_{\underline{n}}],$$

where $\overline{n}$ is the total number of Lagrange nodes, and $\underline{n} = N - \overline{n}$. Coefficients β_i and α_i are then found by imposing the following interpolation conditions at $H(s)$:

$$H(\lambda_i) = w_i, \quad H(\mu_h) = v_h.$$

The interpolation condition at the Lagrange node λ_i is satisfied by setting $\beta_i = \alpha_i w_i$. The interpolation conditions at $\mu_h, h = 1:\underline{n}$, can be written in the matrix form

$$\mathbb{L}\mathbf{a} = \begin{bmatrix} \frac{v_1 - w_1}{\mu_1 - \lambda_1} & \cdots & \frac{v_1 - w_{\overline{n}}}{\mu_1 - \lambda_{\overline{n}}} \\ & \vdots & \\ \frac{v_{\underline{n}} - w_1}{\mu_{\underline{n}} - \lambda_1} & \cdots & \frac{v_{\underline{n}} - w_{\overline{n}}}{\mu_{\underline{n}} - \lambda_{\overline{n}}} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{\overline{n}} \end{bmatrix} = \mathbf{0}, \tag{3.4}$$

where $\mathbb{L}$ is called the Loewner matrix [2]. From (3.4) it follows that the coefficients α_i are contained in the null vector of the Loewner matrix denoted as $\mathbf{a}$.

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Theorem 3.1. *One-variable Lagrange rational function interpolation* [2].

(a) *Given the data set obtained by sampling a rational function $H(s)$ of degree n and partitioning the data such that $\underline{n} \geq n$ and $\overline{n} \geq n$, all possible $\underline{n} \times \overline{n}$ Loewner matrices have rank equal to n.*

(b) *For $\overline{n} = n+1$ and $\underline{n} \geq n$, the Loewner matrix $\mathbb{L}$ has a null vector $\mathbf{a} = \begin{bmatrix}\alpha_1 & \alpha_2 & \dots & \alpha_{n+1}\end{bmatrix}^*, \alpha_i \neq 0$. A unique rational function of minimal degree n which interpolates the sampled set, and therefore coincides with the underlying function $H(s)$, can be obtained in the forms given by (3.3), (3.2), where $\beta_i = w_i \alpha_i, i = 1:n+1$.*

Theorem 3.1 holds for sampled sets that are not degenerate, meaning that none of the sample points is a pole of the underlying rational function. This assumption is justified since in real applications, degenerate sets are rarely encountered due to round-off errors [17].

Once the coefficients α_i are calculated and the rational interpolant $H(s)$ of degree n given by (3.2), (3.3) is obtained, its descriptor system can be easily found.

Theorem 3.2. *Descriptor realization of one-variable rational function $H(s)$.*

(a) *Rational function $H(s) = n(s)/d(s)$ of degree n given with (3.2), (3.3) has the following descriptor realization [4]:*

$$\mathbf{E} = \begin{bmatrix} 1 & -1 & & \\ \vdots & & \ddots & \\ 1 & & & -1 \\ 0 & 0 & \dots & 0 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} \lambda_1 & -\lambda_2 & & \\ \vdots & & \ddots & \\ \lambda_1 & & & -\lambda_{n+1} \\ -\alpha_1 & -\alpha_2 & \dots & -\alpha_{n+1} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \tag{3.5}$$
$$\mathbf{C} = \begin{bmatrix}\beta_1 & \dots & \beta_{n+1}\end{bmatrix}, \qquad \mathbf{D} = \mathbf{0}.$$

(b) *This realization of order $k = n+1$ is C-controllable and C-observable, i.e., minimal, for proper and improper $H(s)$. Otherwise, it is C-controllable and R-observable.*

Remark. This realization is not minimal for strictly proper $H(s)$ since for such functions, $\sum_{i=1}^{n+1} \beta_i = 0$.

Theorem 3.2(b) is original and its proof can be found in Appendix C.1. The procedure for avoiding complex arithmetic and obtaining the realization that has matrices with real entries (when s is complex) is summarized in Appendix A.1.

3.1.2. Identification of low-order descriptor systems. Theorems 3.1 and 3.2 explain how to identify (recover) the minimal descriptor system from its transfer function samples using the Loewner framework. However, the applicability of this framework is much greater. Namely, the Loewner framework can be used for identification of a rational function $\hat{H}(s)$ of low order $\hat{n}$ and its minimal realization, which closely approximates an arbitrary set of transfer function measurements. The procedure is the following. Given the N samples which originate from a rational transfer function of degree n or from a nonrational transfer function, we build (almost) square $\mathbb{L}$ by setting $\overline{n} = N/2$ (for an even number of samples) or $\overline{n} = N/2+1$ (for an odd number of samples). Then by calculating the singular value decomposition of $\mathbb{L}$ we choose a new degree $\hat{n}$ ($\hat{n} < n$ for rational function or $\hat{n} \leq N/2$ for nonrational function samples) such that an $(\hat{n}+1)$th singular value is sufficiently small. Finally, by setting $\overline{n} = \hat{n}+1$ and updating $\mathbb{L}$ accordingly, coefficients $\hat{\alpha}_i, i = 1:\hat{n}$, are obtained from the right singular vector associated

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with the smallest singular value of $\mathbb{L}$. Such coefficients $\hat{\alpha}_i$, together with $\hat{\beta}_i = w_i\hat{\alpha}_i$, form a rational function $\hat{H}(s) = \hat{n}(s)/\hat{d}(s)$ as in (3.2), (3.3) whose descriptor realization $(\hat{\mathbf{E}}, \hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}})$ is given by (3.5). Obtained rational function $\hat{H}(s)$ is of degree $\hat{n}$, and its realization is minimal if and only if $\hat{n}(s)$ and $\hat{d}(s)$ have no common poles, coefficients $\hat{\alpha}_i, i = 1 : \hat{n}+1$, are nonzero, and the highest numerator coefficient $\sum_{i=1}^{\hat{n}+1} \hat{\beta}_i$ is nonzero (see Theorem 3.2 and its proof given in Appendix C.1). In real applications, these conditions are commonly met due to round-off errors. The pointwise approximation error is proportional to the smallest singular value of $\mathbb{L}$ [17], as shown later for the MIMO case (subsection 4.1, Lemma 4.5). Therefore, we can see that the singular value decomposition of $\mathbb{L}$ suggests an appropriate reduced degree $\hat{n}$ of a rational function which closely approximates the sampled data.

3.2. Parametric SISO case.

The results of the two-variable rational interpolation in the Loewner framework from [4] and its extension for the purpose of identifying low-order parametric descriptor systems [17] are presented. The novelty here is derivation of the conditions required for controllability and observability of the constructed parametric descriptor systems.

3.2.1. Two-variable rational function interpolation.

Given the sampled set (2.7) of a parameter-dependent transfer function $\Phi_{ij}(n_y = n_u = 1)$, we seek a two-variable rational function of degree (n, m) (degree n in s and degree m in p) expressed in the Lagrange basis [4, 17],

$$H(s,p) = \frac{n(s,p)}{d(s,p)} = \frac{\sum_{i=1}^{n+1}\sum_{j=1}^{m+1} \beta_{ij} l_{s_i}(s) l_{p_j}(p)}{\sum_{i=1}^{n+1}\sum_{j=1}^{m+1} \alpha_{ij} l_{s_i}(s) l_{p_j}(p)}, \quad \alpha_{ij} \neq 0, \tag{3.6}$$

which interpolates the sampled set, and its minimal realization. Here $l_{p_j}(p) = \prod_{j'=1, j'\neq j}^{m+1}(s - \pi_{j'})$ and $\pi_{j'}$ denote the Lagrange factors and the distinct Lagrange nodes in the parameter variable p. The barycentric form of (3.6) is [4, 17]

$$H(s,p) = \frac{\sum_{i=1}^{n+1}\sum_{j=1}^{m+1} \frac{\beta_{ij}}{(s-\lambda_i)(p-\pi_j)}}{\sum_{i=1}^{n+1}\sum_{j=1}^{m+1} \frac{\alpha_{ij}}{(s-\lambda_i)(p-\pi_j)}}. \tag{3.7}$$

To evaluate the interpolant $H(s,p)$, the sampled data is partitioned into two disjoint sets,

$$\begin{aligned}
&[s_i, \dots, s_N] = [\lambda_1, \dots, \lambda_{\overline{n}}] \cup [\mu_1, \dots, \mu_{\underline{n}}], \\
&[p_1, \dots, p_M] = [\pi_1, \dots, \pi_{\overline{m}}] \cup [\nu_1, \dots, \nu_{\underline{m}}], \\
&[\Phi_{11}, \dots, \Phi_{NM}] = \left[\begin{array}{ccc|ccc}
w_{11} & \dots & w_{1\overline{m}} & \Phi_{1,\overline{m}+1} & \dots & \Phi_{1,M} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
w_{\overline{n}1} & \dots & w_{\overline{n}\overline{m}} & \Phi_{\overline{n},\overline{m}+1} & \dots & \Phi_{\overline{n},M} \\
\hline
\Phi_{\overline{n}+1,1} & \dots & \Phi_{\overline{n}+1,\overline{m}} & v_{11} & \dots & v_{1\underline{m}} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\Phi_{N,1} & \dots & \Phi_{N,\overline{m}} & v_{\underline{n}1} & \dots & v_{\underline{n}\underline{m}}
\end{array}\right] = \mathbf{\Phi},
\end{aligned} \tag{3.8}$$

where $\overline{n}$ and $\overline{m}$ are the total number of Lagrange nodes in s and p and $\underline{n} = N - \overline{n}$, $\underline{m} = M - \overline{m}$. The interpolation condition at the Lagrange node (λ_i, π_j) is satisfied by setting $\beta_{ij} = w_{ij}\alpha_{ij}$.

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Interpolation conditions at nodes (μ_h,ν_d), $h=1:\underline{n}, d=1:\underline{m}$, can be written in the matrix form

$$\mathbb{L}_2\mathbf{a}_2=\begin{bmatrix} c_{1,1}^{1,1} & \dots & c_{1,\overline{m}}^{1,1} & \dots & c_{\overline{n},1}^{1,1} & \dots & c_{\overline{n},\overline{m}}^{1,1} \\ \vdots & & \vdots & & \vdots & & \vdots \\ c_{1,1}^{1,\underline{m}} & \dots & c_{1,\overline{m}}^{1,\underline{m}} & \dots & c_{\overline{n},1}^{1,\underline{m}} & \dots & c_{\overline{n},\overline{m}}^{1,\underline{m}} \\ \vdots & & \vdots & & \vdots & & \vdots \\ c_{1,1}^{\underline{n},1} & \dots & c_{1,\overline{n}}^{\underline{n},1} & \dots & c_{\overline{n},1}^{\underline{n},1} & \dots & c_{\overline{n},\overline{m}}^{\underline{n},1} \\ \vdots & & \vdots & & \vdots & & \vdots \\ c_{1,1}^{\underline{n},\underline{m}} & \dots & c_{1,\overline{m}}^{\underline{n},\underline{m}} & \dots & c_{\overline{n},1}^{\underline{n},\underline{m}} & \dots & c_{\overline{n},\overline{m}}^{\underline{n},\underline{m}} \end{bmatrix}\begin{bmatrix}\alpha_{11} \\ \vdots \\ \alpha_{1\overline{m}} \\ \vdots \\ \alpha_{\overline{n}1} \\ \vdots \\ \alpha_{\overline{n}\overline{m}}\end{bmatrix}=\mathbf{0}, \tag{3.9}$$

where

$$c_{i,j}^{h,d}=\frac{v_{hd}-w_{ij}}{(\mu_h-\lambda_i)(\nu_d-\pi_j)},$$

and $\mathbb{L}_2$ denotes the two-variable Loewner matrix [4].

Theorem 3.3. *Two-variable Lagrange rational function interpolation* [4].

(a) *Given the data set obtained by sampling a two-variable rational function $H(s,p)$ of degrees $(n,\ m)$ with sufficient number of measurements and data partition such that $\overline{n},\underline{n}\geq n$, $\overline{m},\underline{m}\geq m$, all $\underline{nm}\times\overline{nm}$ two-variable Loewner matrices have rank equal to rank $\mathbb{L}_2=\overline{nm}-(\overline{n}-n)(\overline{m}-m)$.*

(b) *For data partition with $(\overline{n},\overline{m})=(n+1,m+1)$, two-variable Loewner matrix $\mathbb{L}_2$ has a null space of dimension one, and its null vector follows the Kronecker structure, $\mathbf{a}_2=\begin{bmatrix}\alpha_{11} & \dots & \alpha_{1m+1} & \dots & \alpha_{n+11} & \dots & \alpha_{n+1m+1}\end{bmatrix}^T, \alpha_{ij}\neq 0$. $H(s,p)$ can then be reconstructed in the forms given by* (3.6), (3.7)*, with $\beta_{ij}=w_i\alpha_{ij}, i=1:n+1, j=1:m+1$.*

Theorem 3.3(a) shows that, unlike the one-variable Loewner matrix $\mathbb{L}$, the two-variable $\mathbb{L}_2$ has the rank dependent on the sample size. Therefore, to reconstruct the two-variable rational function $H(s,p)$ of unknown complexity according to Theorem 3.3(b), we compute the degrees n and m by calculating the maximum rank of all one-variable Loewner matrices associated with each column, $\mathbb{L}(p_j)$, $p_j=const.$, and each row, $\mathbb{L}(s_i)$, $s_i=const.$, of $\mathbf{\Phi}$ (3.8), as suggested in [17].

$$n=\max_j \operatorname{rank}\mathbb{L}(p_j),\quad m=\max_i \operatorname{rank}\mathbb{L}(s_i),\quad j=1:M, i=1:N. \tag{3.10}$$

Theorem 3.4. *Descriptor realization of two-variable rational function $H(s,p)$.*

(a) *Two-variable rational function $H(s,p)$ of degree (n,m) given by* (3.6), (3.7) *has the following descriptor realization* $(\mathbf{E},\mathbf{A}(p),\mathbf{B},\mathbf{C}(p))$ [4]:

$$\mathbf{E}=\begin{bmatrix}1 & -1 & & \\ \vdots & & \ddots & \\ 1 & & & -1 \\ 0 & 0 & \dots & 0\end{bmatrix},\ \mathbf{A}(p)=\begin{bmatrix}\lambda_1 & -\lambda_2 & & \\ \vdots & & \ddots & \\ \lambda_1 & & & -\lambda_{n+1} \\ -\tilde{\alpha}_1(p) & -\tilde{\alpha}_2(p) & \dots & -\tilde{\alpha}_{n+1}(p)\end{bmatrix}, \tag{3.11}$$

$$\mathbf{C}(p)=\begin{bmatrix}\tilde{\beta}_1(p) & \dots & \tilde{\beta}_{n+1}(p)\end{bmatrix},\ \mathbf{B}=\begin{bmatrix}0 & \dots & 0 & 1\end{bmatrix}^*.$$

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where

$$\tilde{\beta}_i(p) = \sum_{j=1}^{m+1} w_{ij}\alpha_{ij} l_{p_j}(p), \quad \tilde{\alpha}_i(p) = \sum_{j=1}^{m+1} \alpha_{ij} l_{p_j}(p). \tag{3.12}$$

(b) *This realization of order $k = n+1$ is C-controllable and C-observable, i.e., minimal if $H(s,p)$ is proper or improper in s for every $p \in \mathbb{C}$; otherwise, it is C-controllable and R-observable.*

Remark. Here, just as in Definitions 2.1 and 2.2, the parametric descriptor system is considered to be controllable and observable if the controllability and observability conditions hold over $p \in \mathbb{C}$. For the realization to be C-controllable over $p \in \mathbb{C}$, $\sum_{i=1}^{n+1} \tilde{\beta}_i(p)$ needs to be a nonzero constant for all $p \in \mathbb{C}$ (see the proof of Theorem 3.4(b) given in Appendix C.2). This is the case for which $H(s,p)$ is proper or improper in s for every $p \in \mathbb{C}$. Otherwise, the system is minimal over $p \in \mathbb{C}\backslash\{r_p\}$, where r_p are zeros of $\sum_{i=1}^{n+1} \tilde{\beta}_i(p)$. Note that the parameters can also be defined on the domain of real numbers, and in that case the conditions are defined for $p \in \mathbb{R}$.

Theorem 3.3 states that the two-variable rational function $H(s,p)$ can be identified from its own samples (assuming the data is sufficiently large) by imposing the interpolation conditions only at points (λ_i, π_j) and (μ_h, ν_d). This is proven in [17] by introducing the generalized two-variable Loewner matrix $\hat{\mathbb{L}}_2$, which includes the whole sampled set, and showing that the null vectors of $\mathbb{L}_2$ and $\hat{\mathbb{L}}_2$ are the same.

3.2.2. Identification of low-order parametric descriptor systems. Analogous to the nonparametric case, the results for two-variable rational interpolation in the Loewner framework can be used for parametric model-order reduction and low-order parametric system identification. Given a general sampled set, the singular value decomposition of the one-variable Loewner matrices $\mathbb{L}(p_j)$ and $\mathbb{L}(s_i)$ as in (3.10) can be used to detect low degrees $(\hat{n}, \hat{m})$ of rational approximant $\hat{H}(s,p)$. $\hat{H}(s,p)$ is constructed by setting $\overline{n} = \hat{n}+1$ and $\overline{m} = \hat{m}+1$ and obtaining $\hat{\alpha}_{ij}$ from the right singular vector of $\hat{\mathbb{L}}_2$ associated with the smallest singular value. As in the nonparametric case, it is justified to expect that the coefficients $\hat{\alpha}_{ij}$ and $\hat{\beta}_{ij}$ form a function $\hat{H}(s,p)$ with degrees $\hat{n}$ and $\hat{m}$ and the nonzero highest numerator coefficient $\sum_{i=1}^{\hat{n}+1} \tilde{\beta}_i(p)$ (polynomial function with zeros r_p). For such a function, descriptor realization (3.11) of order $\hat{k} = \hat{n}+1$ with parametric coefficients of degree $\hat{m}$ is minimal over $p \in \mathbb{C}\backslash\{r_p\}$. It is shown in [17] that the pointwise approximation error is proportional to the smallest singular value of $\hat{\mathbb{L}}_2$. Therefore, this approach allows users to choose appropriate order $\hat{k}$ and complexity $\hat{m}$ in p by tuning $\overline{n}$ and $\overline{m}$.

4. MIMO systems.

First, we generalize the results for the one-variable rational function interpolation problem presented in subsection 3.1 to handle rectangular matrix data. This is referred to as the rational matrix interpolation. These results, combined with the knowledge of two-variable rational function interpolation (subsection 3.2), are then used for further generalization to the two-variable rational matrix interpolation. Both the results of rational matrix interpolation and its parametric counterpart are extended to obtain a new method for identification of low-order MIMO descriptor systems in nonparametric and parametric forms.

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4.1. Nonparametric MIMO systems. Here we present the novel results for one-variable rational matrix interpolation and its application to identification of low-order MIMO descriptor systems with rectangular transfer matrices.

4.1.1. One-variable rational matrix interpolation. Given the set of sampled rectangular transfer matrix $\{s_i, \mathbf{\Phi}_i \mid s_i \in \mathbb{C}, \mathbf{\Phi}_i \in \mathbb{C}^{n_y \times n_u}\}, i = 1 : N$, we seek a rational matrix $\mathbf{H}(s)$ which interpolates the set and its descriptor realization of minimal or as close as possible to minimal order. The considered rational matrix is given by the barycentric formula in the polynomial matrix fraction form

$$\mathbf{H}(s) = \mathbf{N}(s)\mathbf{D}(s)^{-1} = \left(\sum_{i=1}^{\overline{n}} \frac{\boldsymbol{\beta}_i}{s - \lambda_i}\right)\left(\sum_{i=1}^{\overline{n}} \frac{\boldsymbol{\alpha}_i}{s - \lambda_i}\right)^{-1} \text{ for } n_y \geq n_u, \tag{4.1a}$$

$$\mathbf{H}(s) = \mathbf{D}(s)^{-1}\mathbf{N}(s) = \left(\sum_{i=1}^{\overline{n}} \frac{\boldsymbol{\alpha}_i}{s - \lambda_i}\right)^{-1}\left(\sum_{i=1}^{\overline{n}} \frac{\boldsymbol{\beta}_i}{s - \lambda_i}\right) \text{ for } n_y \leq n_u, \tag{4.1b}$$

where det $\boldsymbol{\alpha}_i \neq 0$. Here, $\overline{n}$ denotes the number of Lagrange nodes. Depending on the number of inputs and outputs, the right (4.1a) or left (4.1b) polynomial matrix fraction form is used. This ensures the construction of the coefficient matrices $\boldsymbol{\alpha}_i$ with dimension $n_{min} \times n_{min}$, where $n_{min} = \min(n_u, n_y)$, and allows us to build minimal or close to minimal realization, as explained later.

Lemma 4.1. (a) $\mathbf{H}(s) = \mathbf{N}(s)\mathbf{D}(s)^{-1}$ *given with* (4.1a) *has the degree* $n = (\overline{n} - 1)n_u$ *if and only if* $\mathbf{N}(s)$ *and* $\mathbf{D}(s)$ *are the right coprime polynomial matrices and* $[\mathbf{N}(s)^*, \mathbf{D}(s)^*]^*$ *is given in the column reduced form, i.e.,*

$$\text{rank} \begin{bmatrix} \mathbf{N}(s) \\ \mathbf{D}(s) \end{bmatrix} = n_u \text{ for all finite } s \quad \text{and} \quad \text{rank}\, \mathbf{P}_{hc}\left(\begin{bmatrix} \mathbf{N}(s) \\ \mathbf{D}(s) \end{bmatrix}\right) = n_u,$$

where $\mathbf{P}_{hc}$ *is the highest column coefficient matrix of* $[\mathbf{N}(s)^*, \mathbf{D}(s)^*]^*$*,*

$$\mathbf{P}_{hc}\left(\begin{bmatrix} \mathbf{N}(s) \\ \mathbf{D}(s) \end{bmatrix}\right) = \begin{bmatrix} \sum_{i=1}^{\overline{n}} \boldsymbol{\beta}_i \\ \sum_{i=1}^{\overline{n}} \boldsymbol{\alpha}_i \end{bmatrix}.$$

(b) $\mathbf{H}(s) = \mathbf{D}(s)^{-1}\mathbf{N}(s)$ *given by* (4.1b) *has the degree* $n = (\overline{n} - 1)n_y$ *if and only if* $\mathbf{N}(s)$ *and* $\mathbf{D}(s)$ *are the left coprime polynomial matrices and* $[\mathbf{N}(s), \mathbf{D}(s)]$ *is given in the row reduced form, i.e.,*

$$\text{rank} \begin{bmatrix} \mathbf{N}(s) & \mathbf{D}(s) \end{bmatrix} = n_y \text{ for all finite } s \quad \text{and} \quad \text{rank}\, \mathbf{P}_{hr}\left(\begin{bmatrix} \mathbf{N}(s) & \mathbf{D}(s) \end{bmatrix}\right) = n_y,$$

where $\mathbf{P}_{hr}([\mathbf{N}(s), \mathbf{D}(s)]) = [\sum_{i=1}^{\overline{n}} \boldsymbol{\beta}_i, \sum_{i=1}^{\overline{n}} \boldsymbol{\alpha}_i]$ *denotes the highest row coefficient matrix of* $[\mathbf{N}(s), \mathbf{D}(s)]$.

Under the conditions stated in Lemma 4.1, $\mathbf{H}(s)$ obtains the highest degree possible. Therefore, the degree n of $\mathbf{H}(s)$ as in (4.1a), (4.1b) is bounded from above,

$$n \leq (\overline{n} - 1)n_{min}.$$

Lemma 4.1 relies on the auxiliary lemma of [3].

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The sampled data is partitioned into two disjoint sets, $\{\lambda_i, \mathbf{W}_i, i=1:\overline{n}\}$ and $\{\mu_h, \mathbf{V}_h, h=1:\underline{n}\}$. The coefficient matrices $\boldsymbol{\beta}_i$ and $\boldsymbol{\alpha}_i$ are found by imposing the interpolation conditions on $\mathbf{H}(s)$. Interpolation conditions at the Lagrange nodes λ_i are satisfied by setting $\boldsymbol{\beta}_i = \mathbf{W}_i\boldsymbol{\alpha}_i$ in (4.1a) and $\boldsymbol{\beta}_i = \boldsymbol{\alpha}_i\mathbf{W}_i$ in (4.1b). To solve for the coefficients $\boldsymbol{\alpha}_i$, we introduce the right and left MIMO Loewner matrices,[1] $\mathbb{L}^r$ and $\mathbb{L}^l$, such that

$$\mathbb{L}^r\mathbb{A}^r = \begin{bmatrix} \frac{\mathbf{V}_1-\mathbf{W}_1}{\mu_1-\lambda_1} & \cdots & \frac{\mathbf{V}_1-\mathbf{W}_{\overline{n}}}{\mu_1-\lambda_{\overline{n}}} \\ & \vdots & \\ \frac{\mathbf{V}_{\underline{n}}-\mathbf{W}_1}{\mu_{\underline{n}}-\lambda_{\overline{n}}} & \cdots & \frac{\mathbf{V}_{\underline{n}}-\mathbf{W}_{\overline{n}}}{\mu_{\underline{n}}-\lambda_{\overline{n}}} \end{bmatrix} \begin{bmatrix} \boldsymbol{\alpha}_1 \\ \vdots \\ \boldsymbol{\alpha}_{\overline{n}} \end{bmatrix} = \mathbf{0} \text{ for } n_y \geq n_u, \tag{4.2a}$$

$$\mathbb{L}^l\mathbb{A}^l = \begin{bmatrix} \frac{\mathbf{V}_1^*-\mathbf{W}_1^*}{\mu_1-\lambda_1} & \cdots & \frac{\mathbf{V}_1^*-\mathbf{W}_{\overline{n}}^*}{\mu_1-\lambda_{\overline{n}}} \\ & \vdots & \\ \frac{\mathbf{V}_{\underline{n}}^*-\mathbf{W}_1^*}{\mu_{\underline{n}}-\lambda_1} & \cdots & \frac{\mathbf{V}_{\underline{n}}^*-\mathbf{W}_{\overline{n}}^*}{\mu_{\underline{n}}-\lambda_{\overline{n}}} \end{bmatrix} \begin{bmatrix} \boldsymbol{\alpha}_1^* \\ \vdots \\ \boldsymbol{\alpha}_{\overline{n}}^* \end{bmatrix} = \mathbf{0} \text{ for } n_y \leq n_u. \tag{4.2b}$$

The right and left MIMO Loewner matrices built from samples of some rational matrix $\mathbf{R}(s)$ of degree n with the minimal descriptor realization $(\bar{\mathbf{E}}, \bar{\mathbf{A}}, \bar{\mathbf{B}}, \bar{\mathbf{C}})$,

$$\mathbb{L}^r_{hi} = \frac{\bar{\mathbf{C}}\left[(\mu_h\bar{\mathbf{E}}-\bar{\mathbf{A}})^{-1} - (\lambda_i\bar{\mathbf{E}}-\bar{\mathbf{A}})^{-1}\right]\bar{\mathbf{B}}}{\mu_h-\lambda_i},$$
$$\mathbb{L}^l_{hi} = \frac{\bar{\mathbf{B}}^*\left[\left((\mu_h\bar{\mathbf{E}}-\bar{\mathbf{A}})^{-1}\right)^* - \left((\lambda_i\bar{\mathbf{E}}-\bar{\mathbf{A}})^{-1}\right)^*\right]\bar{\mathbf{C}}^*}{\mu_h-\lambda_i},$$

can be factored as the product of the MIMO generalized observability matrix $\mathcal{O}$, matrix $\bar{\mathbf{E}}$, and the MIMO generalized controllability $\mathcal{C}$,

$$\mathbb{L}^r = -\mathcal{O}^r\bar{\mathbf{E}}\mathcal{C}^r, \quad \mathbb{L}^l = -\mathcal{C}^{l*}\bar{\mathbf{E}}^*\mathcal{O}^{l*}, \tag{4.3}$$

where

$$\mathcal{O}^r = \begin{bmatrix} \bar{\mathbf{C}}(\mu_1\bar{\mathbf{E}}-\bar{\mathbf{A}})^{-1} \\ \vdots \\ \bar{\mathbf{C}}(\mu_{\underline{n}}\bar{\mathbf{E}}-\bar{\mathbf{A}})^{-1} \end{bmatrix}, \quad \mathcal{C}^r = \left[(\lambda_1\bar{\mathbf{E}}-\bar{\mathbf{A}})^{-1}\bar{\mathbf{B}} \quad \ldots \quad (\lambda_{\overline{n}}\bar{\mathbf{E}}-\bar{\mathbf{A}})^{-1}\bar{\mathbf{B}}\right],$$

$$\mathcal{O}^l = \begin{bmatrix} \bar{\mathbf{C}}(\lambda_1\bar{\mathbf{E}}-\bar{\mathbf{A}})^{-1} \\ \vdots \\ \bar{\mathbf{C}}(\lambda_{\overline{n}}\bar{\mathbf{E}}-\bar{\mathbf{A}})^{-1} \end{bmatrix}, \quad \mathcal{C}^l = \left[(\mu_1\bar{\mathbf{E}}-\bar{\mathbf{A}})^{-1}\bar{\mathbf{B}} \quad \ldots \quad (\mu_{\underline{n}}\bar{\mathbf{E}}-\bar{\mathbf{A}})^{-1}\bar{\mathbf{B}}\right].$$

Under the minimality assumption, rank $\bar{\mathbf{E}} = n$ [37].

Lemma 4.2. *Let* $(\bar{\mathbf{E}}, \bar{\mathbf{A}}, \bar{\mathbf{B}})$ *and* $(\bar{\mathbf{E}}, \bar{\mathbf{A}}, \bar{\mathbf{C}})$ *be a controllable and observable triple of minimal order* k_{min^-}, *and assume* $\lambda_i, \mu_h, i=1:\overline{n}, h=1:\underline{n}$ *are not generalized eigenvalues of* $(\bar{\mathbf{A}}, \bar{\mathbf{E}})$. *Then the rank of the MIMO generalized controllability and observability matrix is* k_{min}. *That is,*

[1] Note that these matrices differ from the left and right Loewner matrices introduced in [32].

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$$\text{rank}\,\mathcal{O}^r = k_{min},\ \text{rank}\,\mathcal{C}^r = k_{min},\ \text{for}\, n_y \geq n_u, \tag{4.4a}$$

$$\text{rank}\,\mathcal{O}^l = k_{min},\ \text{rank}\,\mathcal{C}^l = k_{min},\ \text{for}\, n_y \leq n_u, \tag{4.4b}$$

provided that $\overline{n} > n$, $\underline{n} > n$.

Proof of Lemma 4.2 is given in Appendix C.3. An immediate consequence of the factorization (4.3) and Lemma 4.2 is the following theorem.

Theorem 4.3. *If sampled data originates from a rational matrix* $\mathbf{R}(s)$ *of (McMillan) degree* n *which has a minimal realization* $(\bar{\mathbf{E}}, \bar{\mathbf{A}}, \bar{\mathbf{B}}, \bar{\mathbf{C}})$ *of order* k_{min}, $\mathbf{R}(s) = \bar{\mathbf{C}}(s\bar{\mathbf{E}} - \bar{\mathbf{A}})^{-1}\bar{\mathbf{B}}$, *then every MIMO Loewner matrix which has more than* n *block columns,* $\overline{n} > n$, *and more than* n *block rows,* $\underline{n} > n$, *has a rank equal to* n.

From Theorem 4.3, we know the rank of the MIMO Loewner matrix $\mathbb{L}^{r/l}$, constructed from samples of a rational matrix of degree n when $\overline{n} > n$. When $\overline{n} = n+1$, the null space of $\mathbb{L}^{r/l}$ is of dimension $(n+1)n_{min} - n$. Therefore, under the condition of full-rank coefficient matrices $\boldsymbol{\alpha}_i$, the sampled rational matrix is reconstructed as (4.1) from the null space of the MIMO Loewner matrix. Even though this condition is often met, it is not always guaranteed to hold.

Theorem 4.4.

(a) *The descriptor realization of rational matrix* $\mathbf{H}(s)$ *of degree* n *given as right polynomial matrix fraction* (4.1a) *is*

$$\mathbf{E} = \begin{bmatrix} \mathbf{I}_{n_u} & -\mathbf{I}_{n_u} & & \\ \vdots & & \ddots & \\ \mathbf{I}_{n_u} & & & -\mathbf{I}_{n_u} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} \lambda_1\mathbf{I}_{n_u} & -\lambda_2\mathbf{I}_{n_u} & & \\ \vdots & & \ddots & \\ \lambda_1\mathbf{I}_{n_u} & & & -\lambda_{\overline{n}}\mathbf{I}_{n_u} \\ -\boldsymbol{\alpha}_1 & -\boldsymbol{\alpha}_2 & \dots & -\boldsymbol{\alpha}_{\overline{n}} \end{bmatrix},$$
$$\mathbf{C} = \begin{bmatrix} \boldsymbol{\beta}_1 & \dots & \boldsymbol{\beta}_{\overline{n}} \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} \mathbf{0} & \dots & \mathbf{0} & \mathbf{I}_{n_u} \end{bmatrix}^*, \tag{4.5}$$

while the descriptor realization of $\mathbf{H}(s)$ *of degree* n *given as left polynomial matrix fraction* (4.1b) *is*

$$\mathbf{E} = \begin{bmatrix} \mathbf{I}_{n_y} & \mathbf{I}_{n_y} & \dots & \mathbf{0} \\ -\mathbf{I}_{n_y} & & & \mathbf{0} \\ & \ddots & & \\ & & -\mathbf{I}_{n_y} & \mathbf{0} \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} \lambda_1\mathbf{I}_{n_y} & \lambda_1\mathbf{I}_{n_y} & \dots & -\boldsymbol{\alpha}_1 \\ -\lambda_2\mathbf{I}_{n_y} & & & -\boldsymbol{\alpha}_2 \\ & \ddots & & \\ & & -\lambda_{\overline{n}}\mathbf{I}_{n_y} & -\boldsymbol{\alpha}_{\overline{n}} \end{bmatrix},$$
$$\mathbf{C} = \begin{bmatrix} \mathbf{0} & \dots & \mathbf{0} & \mathbf{I}_{n_y} \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} \boldsymbol{\beta}_1^* & \dots & \boldsymbol{\beta}_{\overline{n}}^* \end{bmatrix}^*. \tag{4.6}$$

(b) *Realization* (4.5) *is C-controllable and R-observable if and only if* $\mathbf{N}(s)$ *and* $\mathbf{D}(s)$ *are the right coprime polynomial matrices. Realization* (4.6) *is C-observable and R-controllable if and only if* $\mathbf{N}(s)$ *and* $\mathbf{D}(s)$ *are the left coprime polynomial matrices.*

(c) *Realization* (4.5) *is minimal if and only if the order* k *is* $k = k_{min} = n + n_u$ *and* $\sum_{i=1}^{\overline{n}} \boldsymbol{\beta}_i$ *has full column rank. Realization* (4.6) *is minimal if and only if the order* k *is* $k = k_{min} = n + n_y$ *and* $\sum_{i=1}^{\overline{n}} \boldsymbol{\beta}_i$ *has full row rank.*

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Remark. Notice that the introduction of the left polynomial fraction description of $\mathbf{H}(s)$ as in (4.1b) is necessary for construction of controllable and observable realizations when $n_y < n_u$. Furthermore, notice that both (4.5) and (4.6) cannot be minimal for strictly proper $\mathbf{H}(s)$.

According to Theorem 4.4 (see proof in Appendix C.4), realization is minimal only if $\mathbf{H}(s)$ has the maximum degree, $n = (\overline{n} - 1)n_{min}$. Therefore, in an attempt to recover a rational matrix of some degree n in the minimal realization form, we set the number of Lagrange nodes as

$$\overline{n} = n/n_{min} + 1.$$

This, however, is only possible if n is divisible by n_{min}. Otherwise, to obtain the rational matrix of order n, $\overline{n} > n/n_{min} + 1$ and therefore $k > n + n_{min} = k_{min}$. Moreover, even if n is divisible by n_{min}, the underlying rational function cannot always be recovered by calculating the null vector of the MIMO Loewner matrix. This is a consequence of Theorem 4.3 which states that the MIMO Loewner matrix is guaranteed to detect the degree of the underlying rational matrix only if $\overline{n} > n$. Thus, if $\overline{n} = n/n_{min} + 1 < n$, the null space $\mathbb{A}$ can be of dimension smaller than n_{min}. Therefore, the presented methodology cannot always recover the rational matrix of order n in the minimal realization form from its own samples. The limitation regarding minimality of constructed descriptor systems, however, tends to diminish when the proposed method is used for reduced order modeling and low-order system identification purposes.

4.1.2. Identification of low-order descriptor systems. Given an arbitrary sampled set of rectangular transfer matrices, we seek the rational matrix $\hat{\mathbf{H}}(s)$,

$$\hat{\mathbf{H}}(s) = \left(\sum_{i=1}^{\overline{n}} \frac{\mathbf{W}_i \hat{\boldsymbol{\alpha}}_i}{s - \lambda_i}\right)\left(\sum_{i=1}^{\overline{n}} \frac{\hat{\boldsymbol{\alpha}}_i}{s - \lambda_i}\right)^{-1} \text{ for } n_y \geq n_u, \tag{4.7a}$$

$$\hat{\mathbf{H}}(s) = \left(\sum_{i=1}^{\overline{n}} \frac{\hat{\boldsymbol{\alpha}}_i}{s - \lambda_i}\right)^{-1}\left(\sum_{i=1}^{\overline{n}} \frac{\hat{\boldsymbol{\alpha}}_i \mathbf{W}_i}{s - \lambda_i}\right) \text{ for } n_y \leq n_u, \tag{4.7b}$$

such that it accurately approximates the sampled data and its minimal descriptor realization. First, we determine the desired low degree $\hat{n}$ of $\hat{\mathbf{H}}(s)$, for which the data can be accurately approximated. This can be done by calculating the singular value decomposition of the Loewner matrix based on tangential interpolation from [25], as suggested in [23]. Degree $\hat{n}$ is chosen such that it is divisible by n_{min} and the associated singular value is sufficiently small. Then we partition the data such that $\overline{n} = \hat{n}/n_{min} + 1$. With this setting, both rational and nonrational matrix samples usually form numerically full-rank MIMO Loewner matrices. Coefficients $\hat{\boldsymbol{\alpha}}_i$ are thus obtained from the right singular vectors of $\mathbb{L}^{r/l}$ associated with the n_{min} smallest singular values. In real applications, due to round-off errors, $\hat{\boldsymbol{\alpha}}_i$ are full-rank matrices, and the rational approximant $\hat{\mathbf{H}}(s)$ tends to obtain the desired degree $\hat{n} = (\overline{n} - 1)n_{min}$, which is the maximum degree for the given $\overline{n}$. Furthermore, for the same reason, $\sum_{i=1}^{\overline{n}} \mathbf{W}_i \hat{\boldsymbol{\alpha}}_i$ and $\sum_{i=1}^{\overline{n}} \hat{\boldsymbol{\alpha}}_i \mathbf{W}_i$ are expected to have full column and full row rank, respectively. As $\overline{n}$ decreases, the chances of these assumptions failing are lower. Therefore, it is reasonable to expect that the descriptor realization of $\hat{\mathbf{H}}(s)$, which can be obtained from Theorem 4.4, is C-observable and C-controllable.

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Rational matrix $\hat{\mathbf{H}}(s)$ as in (4.7) interpolates the sampled set at the Lagrange nodes by definition, under the condition $\det \hat{\boldsymbol{\alpha}}_i \neq 0$. Thus, the pointwise approximation error matrix is evaluated at nodes μ_h,

$$\mathbf{E}_{rr}(\mu_h) = \mathbf{V}_h - \hat{\mathbf{H}}(\mu_h) = \left(\sum_{i=1}^{\overline{n}} \frac{\mathbf{V}_h - \mathbf{W}_i}{\mu_h - \lambda_i}\hat{\boldsymbol{\alpha}}_i\right)\left(\sum_{i=1}^{\overline{n}} \frac{\hat{\boldsymbol{\alpha}}_i}{\mu_h - \lambda_i}\right)^{-1} \text{ for } n_y \geq n_u$$

$$\mathbf{E}^*_{rr}(\mu_h) = \mathbf{V}^*_h - \hat{\mathbf{H}}^*(\mu_h) = \left(\sum_{i=1}^{\overline{n}} \frac{\mathbf{V}^*_h - \mathbf{W}^*_i}{\mu_h - \lambda_i}\hat{\boldsymbol{\alpha}}^*_i\right)\left(\sum_{i=1}^{\overline{n}} \frac{\hat{\boldsymbol{\alpha}}^*_i}{\mu_h - \lambda_i}\right)^{-1} \text{ for } n_y \leq n_u.$$

Here the first term represents the hth block row of $\mathbb{L}^r$ and $\mathbb{L}^l$, respectively. We can see that when the coefficients $\hat{\boldsymbol{\alpha}}_i$ are obtained from the null space of $\mathbb{L}^{r/l}$, the pointwise error is zero.

Lemma 4.5. *Let $\hat{\boldsymbol{\Sigma}} = \mathrm{diag}\,(\hat{\sigma}_1, \ldots, \hat{\sigma}_{n_{min}})$ be the matrix of the n_{min} smallest singular values of $\mathbb{L}^{r/l}$, and let $\hat{\mathbf{X}}, \hat{\mathbf{Y}}$ be the matrices of the associated n_{min} right and left singular vectors, $\hat{\mathbf{Y}} = [\hat{\mathbf{Y}}^*_1, \ldots, \hat{\mathbf{Y}}^*_{\underline{n}}]^*$, $\hat{\mathbf{X}} = [\hat{\mathbf{X}}^*_1, \ldots, \hat{\mathbf{X}}^*_{\overline{n}}]^*$. Then, by setting $\hat{\mathbb{A}}^{r/l} = \hat{\mathbf{X}}$, the pointwise approximation error matrix is given as*

$$\mathbf{E}_{rr}(\mu_h) = \mathbf{V}_h - \hat{\mathbf{H}}(\mu_h) = \hat{\mathbf{Y}}_h\hat{\boldsymbol{\Sigma}}\left(\sum_{i=1}^{\overline{n}} \frac{\hat{\boldsymbol{\alpha}}_i}{\mu_h - \lambda_i}\right)^{-1}, \ h = 1:\underline{n} \text{ for } n_y \geq n_u,$$

$$\mathbf{E}^*_{rr}(\mu_h) = \mathbf{V}^*_h - \hat{\mathbf{H}}^*(\mu_h) = \hat{\mathbf{Y}}_h\hat{\boldsymbol{\Sigma}}\left(\sum_{i=1}^{\overline{n}} \frac{\hat{\boldsymbol{\alpha}}^*_i}{\mu_h - \lambda_i}\right)^{-1}, \ h = 1:\underline{n} \text{ for } n_y \leq n_u.$$

According to Lemma 4.5, the pointwise error of the rational approximant $\hat{\mathbf{H}}(s)$ is proportional to the smallest singular values of $\mathbb{L}^{r/l}$. More precisely, each entry of the error matrix $\mathbf{E}_{rr}(\mu_h)$ can be expressed as a linear combination of the n_{min} smallest singular values. Notice that, when the sampled data originates from a rational matrix of order n, setting $\overline{n} > n$ leads to zero error matrix $\mathbf{E}_{rr}(s)$ at nodes $s = \mu_h$ since $\hat{\boldsymbol{\Sigma}} = \mathbf{0}$ (according to Theorem 4.3). The sampled data is therefore fully interpolated if obtained $\boldsymbol{\alpha}_i$ are full-rank. Thus by tuning $\overline{n}$, we can either construct rational interpolant $\mathbf{H}(s)$ or approximant $\hat{\mathbf{H}}(s)$.

4.2. Parametric MIMO systems.

Here we generalize the results for the one-variable rational matrix interpolation problem to solve the two-variable rational matrix interpolation problem. We also present the novel method for identification of the parametric MIMO descriptor systems of low order, which is based on these results.

4.2.1. Two-variable rational matrix interpolation.

Given the sampled set of parameter-dependent transfer matrix (2.7), we seek a two-variable rational matrix $\mathbf{H}(s,p)$ given in the Lagrange basis,

$$\mathbf{H}(s,p) = \mathbf{N}(s,p)\mathbf{D}(s,p)^{-1} \text{ for } n_y \geq n_u, \tag{4.8a}$$

$$\mathbf{H}(s,p) = \mathbf{D}(s,p)^{-1}\mathbf{N}(s,p) \text{ for } n_y \leq n_u, \tag{4.8b}$$

$$\mathbf{D}(s,p) = \sum_{i=1}^{\overline{n}}\sum_{j=1}^{\overline{m}} \frac{\boldsymbol{\beta}_{ij}}{(s-\lambda_i)(p-\pi_j)}, \ \mathbf{N}(s,p) = \sum_{i=1}^{\overline{n}}\sum_{j=1}^{\overline{m}} \frac{\boldsymbol{\alpha}_{ij}}{(s-\lambda_i)(p-\pi_j)}, \tag{4.8c}$$

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and $\det \boldsymbol{\alpha}_{ij} \neq 0$, which interpolates the set, and its minimal (or close to minimal) realization. Degrees in s and p of $\mathbf{H}(s,p)$ as in (4.8) are upper bounded,

$$n \leq (\overline{n}-1)n_{min} \quad \text{and} \quad m \leq (\overline{m}-1)n_{min}.$$

$\mathbf{H}(s,p)$ obtains the maximum degree in variables s and p when the conditions stated in Lemma 4.1 hold for a fixed p and a fixed s, respectively. This is stated in Lemma 4.6.

Lemma 4.6. *$\mathbf{H}(s,p) = \mathbf{N}(s,p)\mathbf{D}(s,p)^{-1}$ given by (4.8) has degree $n = (\overline{n}-1)n_u$ in s and $m = (\overline{m}-1)n_u$ in p if and only if $\mathbf{N}(s,p)$ and $\mathbf{D}(s,p)$ are the right coprime two-variable polynomial matrices,*

$$\text{(4.9)} \qquad rank \begin{bmatrix} \mathbf{N}(s,p) \\ \mathbf{D}(s,p) \end{bmatrix} = n_u \text{ for all finite } s,p \in \mathbb{C};$$

$[\mathbf{N}(s,p)^*, \mathbf{D}(s,p)^*]^*$ is given in the column reduced form for some $p = const.$, i.e., the highest column coefficient matrix with respect to variable s is full column rank,

$$\text{(4.10a)} \qquad rank\, \mathbf{P}_{hc}^s \left(\begin{bmatrix} \mathbf{N}(s,p) \\ \mathbf{D}(s,p) \end{bmatrix} \right) = rank \begin{bmatrix} \sum_{i=1}^{\overline{n}} \tilde{\boldsymbol{\beta}}_i(p) \\ \sum_{i=1}^{\overline{n}} \tilde{\boldsymbol{\alpha}}_i(p) \end{bmatrix} = n_u \text{ for some } p = const.,$$

$$\text{(4.10b)} \qquad \tilde{\boldsymbol{\beta}}_i(p) = \sum_{j=1}^{\overline{m}} \boldsymbol{\beta}_{ij} l_{p_j}(p),\ \tilde{\boldsymbol{\alpha}}_i(p) = \sum_{j=1}^{\overline{m}} \boldsymbol{\alpha}_{ij} l_{p_j}(p);$$

and $[\mathbf{N}(s,p)^*, \mathbf{D}(s,p)^*]^*$ is given in the column reduced form for some $s = const.$ i.e., the highest column coefficient matrix with respect to variable p is full column rank,

$$rank\, \mathbf{P}_{hc}^p \left(\begin{bmatrix} \mathbf{N}(s,p) \\ \mathbf{D}(s,p) \end{bmatrix} \right) = rank \begin{bmatrix} \sum_{j=1}^{\overline{m}} \tilde{\boldsymbol{\beta}}_j(s) \\ \sum_{j=1}^{\overline{m}} \tilde{\boldsymbol{\alpha}}_j(s) \end{bmatrix} = n_u \text{ for some } s = const.,$$

$$\tilde{\boldsymbol{\beta}}_j(s) = \sum_{i=1}^{\overline{n}} \boldsymbol{\beta}_{ij} l_{s_i}(s),\ \tilde{\boldsymbol{\alpha}}_j(s) = \sum_{i=1}^{\overline{n}} \boldsymbol{\alpha}_{ij} l_{s_i}(s).$$

The conditions for $\mathbf{H}(s,p) = \mathbf{D}(s,p)^{-1}\mathbf{N}(s,p)$ given by (4.8) can be analogously derived by applying conditions for the one-variable rational matrix from Lemma 4.1 while keeping one variable fixed.

Sampled data is partitioned as in the SISO case (3.8). The interpolation conditions at the Lagrange points, $\mathbf{H}(\lambda_i,\pi_j) = \mathbf{W}_{ij}$, are satisfied by setting $\boldsymbol{\beta}_{ij} = \mathbf{W}_{ij}\boldsymbol{\alpha}_{ij}$ for $n_y \geq n_u$ and $\boldsymbol{\beta}_{ij} = \boldsymbol{\alpha}_{ij}\mathbf{W}_{ij}$ for $n_y \leq n_u$. Parameters $\boldsymbol{\alpha}_{ij}$ are determined by imposing additional interpolation conditions on $\mathbf{H}(s,p)$. Interpolation conditions $\mathbf{H}(\mu_h,\nu_d) = \mathbf{V}_{hd}$ at the nodes (μ_h,ν_d), $h = 1:\underline{n}, d = 1:\underline{m}$, can be expressed in the following matrix form:

$$\mathbb{L}_2^r \mathbb{A}_2^r = \mathbf{0} \text{ for } n_y \geq n_u, \quad \mathbb{L}_2^l \mathbb{A}_2^l = \mathbf{0} \text{ for } n_y \leq n_u.$$

Here, the two-variable MIMO Loewner matrices $\mathbb{L}_2^r$ and $\mathbb{L}_2^l$ have the same structure as in (3.9), with matrix-valued entries,

$$\text{(4.11)} \qquad [c^r]_{i,j}^{h,d} = \frac{\mathbf{V}_{hd} - \mathbf{W}_{ij}}{(\mu_h - \lambda_i)(\nu_d - \pi_j)},\ [c^l]_{i,j}^{h,d} = \frac{\mathbf{V}_{hd}^* - \mathbf{W}_{ij}^*}{(\mu_h - \lambda_i)(\nu_d - \pi_j)}.$$

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The null vectors of $\mathbb{L}_2^r$ and $\mathbb{L}_2^l$ are given as

$$\begin{aligned}\mathbb{A}_2^r &= \begin{bmatrix}\boldsymbol{\alpha}_{11}^* & \dots & \boldsymbol{\alpha}_{1\overline{m}}^* & \dots & \boldsymbol{\alpha}_{\overline{n}1}^* & \dots & \boldsymbol{\alpha}_{\overline{nm}}^*\end{bmatrix}^*,\\ \mathbb{A}_2^l &= \begin{bmatrix}\boldsymbol{\alpha}_{11} & \dots & \boldsymbol{\alpha}_{1\overline{m}} & \dots & \boldsymbol{\alpha}_{\overline{n}1} & \dots & \boldsymbol{\alpha}_{\overline{nm}}\end{bmatrix}^*.\end{aligned} \tag{4.12}$$

Analogously to the SISO theory presented in [17], we introduce matrices $\mathbb{L}_{\lambda_i}^{r/l}$ and $\mathbb{L}_{\pi_j}^{r/l}$, one-variable MIMO Loewner matrices with the block matrix structure associated with the ith row and the jth column of $\mathbf{\Phi}$ (3.8), respectively. Interpolation conditions at nodes (λ_i, ν_d), for a fixed λ_i and $d = 1:\underline{m}$ can then be written as

$$\mathbb{L}_{\lambda_i}^r \mathbb{A}_{\lambda_i}^r = \begin{bmatrix}\frac{\mathbf{\Phi}_{i,\overline{m}+1}-\mathbf{W}_{i1}}{\nu_1-\pi_1} & \dots & \frac{\mathbf{\Phi}_{i,\overline{m}+1}-\mathbf{W}_{i\overline{m}}}{\nu_1-\pi_{\overline{m}}}\\ \vdots & \ddots & \vdots\\ \frac{\mathbf{\Phi}_{i,M}-\mathbf{W}_{i1}}{\nu_{\underline{n}}-\pi_1} & \dots & \frac{\mathbf{\Phi}_{i,M}-\mathbf{W}_{i\overline{m}}}{\nu_{\underline{n}}-\pi_{\overline{m}}}\end{bmatrix}\begin{bmatrix}\boldsymbol{\alpha}_{\lambda_{i1}}\\ \vdots\\ \boldsymbol{\alpha}_{\lambda_{i\overline{m}}}\end{bmatrix} = \mathbf{0} \text{ for } n_y \geq n_u,$$

while the interpolation conditions at nodes (μ_h, π_j), for a fixed π_j and $h = 1:\underline{n}$ are given as

$$\mathbb{L}_{\pi_j}^r \mathbb{A}_{\pi_j}^r = \begin{bmatrix}\frac{\mathbf{\Phi}_{\overline{n}+1,j}-\mathbf{W}_{1j}}{\mu_1-\lambda_1} & \dots & \frac{\mathbf{\Phi}_{\overline{n}+1,j}-\mathbf{W}_{\overline{n}j}}{\mu_1-\lambda_{\overline{n}}}\\ \vdots & \ddots & \vdots\\ \frac{\mathbf{\Phi}_{N,j}-\mathbf{W}_{1j}}{\mu_{\underline{n}}-\lambda_1} & \dots & \frac{\mathbf{\Phi}_{N,j}-\mathbf{W}_{\overline{n}j}}{\mu_{\underline{n}}-\lambda_{\overline{n}}}\end{bmatrix}\begin{bmatrix}\boldsymbol{\alpha}_{\pi_{1j}}\\ \vdots\\ \boldsymbol{\alpha}_{\pi_{\overline{n}j}}\end{bmatrix} = \mathbf{0} \text{ for } n_y \geq n_u.$$

If $n_y \leq n_u$, the left counterparts of $\mathbb{L}_{\lambda_i}^r$ and $\mathbb{L}_{\pi_j}^r$, obtained by plugging the transpose values of the sampled data, $\mathbb{L}_{\lambda_i}^l(\mathbf{\Phi}) = \mathbb{L}_{\lambda_i}^r(\mathbf{\Phi}^*)$, $\mathbb{L}_{\pi_j}^l(\mathbf{\Phi}) = \mathbb{L}_{\pi_j}^r(\mathbf{\Phi}^*)$, are used. In this case, the associated null vectors also contain transpose of coefficients. Finally, we can write all the interpolation conditions, which include the whole sampled set, using a single matrix $\hat{\mathbb{L}}_2$,

$$\hat{\mathbb{L}}_2^r \hat{\mathbb{A}}_2^r = \begin{bmatrix}\mathbb{L}_\lambda^r\\ \mathbb{L}_\pi^r\\ \mathbb{L}_2^r\end{bmatrix}\hat{\mathbb{A}}_2^r = \mathbf{0} \text{ for } n_y \geq n_u, \quad \hat{\mathbb{L}}_2^l \hat{\mathbb{A}}_2^l = \begin{bmatrix}\mathbb{L}_\lambda^l\\ \mathbb{L}_\pi^l\\ \mathbb{L}_2^l\end{bmatrix}\hat{\mathbb{A}}_2^l = \mathbf{0} \text{ for } n_y \leq n_u, \tag{4.13}$$

where

$$\mathbb{L}_\lambda^{r/l} = \begin{bmatrix}\mathbb{L}_{\lambda_1}^{r/l} & & \\ & \ddots & \\ & & \mathbb{L}_{\lambda_{\overline{n}}}^{r/l}\end{bmatrix},$$

$$\mathbb{L}_\pi^{r/l} = \left[\begin{array}{ccc|c|ccc}\mathbb{L}_{\pi_1}^{r/l}(:,1) & & & & \mathbb{L}_{\pi_1}^{r/l}(:,\overline{n}) & & \\ & \ddots & & \dots & & \ddots & \\ & & \mathbb{L}_{\pi_{\overline{m}}}^{r/l}(:,1) & & & & \mathbb{L}_{\pi_{\overline{m}}}^{r/l}(:,\overline{n})\end{array}\right].$$

Theorem 4.7. *If the sampled data is obtained by sampling a two-variable rational matrix, it follows that*

$$\textit{rank}\ \mathbb{L}_2^{r/l} = \textit{rank}\begin{bmatrix}\mathbb{L}_\lambda^{r/l}\\ \mathbb{L}_\pi^{r/l}\end{bmatrix} = \textit{rank}\ \hat{\mathbb{L}}_2^{r/l} \textit{ and } \textit{Ker}\ \mathbb{L}_2^{r/l} = \textit{Ker}\begin{bmatrix}\mathbb{L}_\lambda^{r/l}\\ \mathbb{L}_\pi^{r/l}\end{bmatrix} = \textit{Ker}\ \hat{\mathbb{L}}_2^{r/l}.$$

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According to Theorem 4.7, the null spaces of $\mathbb{L}_2^{r/l}$ and $\hat{\mathbb{L}}_2^{r/l}$ are equal when the samples originate from a two-variable rational matrix. These null spaces, according to the proof of Theorem 4.7 given in Appendix C.5, can be formed by multiplying the null vectors of the one-variable MIMO Loewner matrices, as demonstrated in [4] for the SISO case. Therefore, taking into account Theorem 4.3, we can conclude that for the sufficient amount of data ($\overline{n} > n, \overline{m} > m$) there will exist a null space of $\mathbb{L}_2^{r/l}$ and $\hat{\mathbb{L}}_2^{r/l}$. Under the condition of full-rank coefficients $\boldsymbol{\alpha}_{ij}$, the sampled rational matrix is reconstructed in the barycentric form (4.8) from the null space of the two-variable MIMO Loewner matrix.

Theorem 4.8.

(a) *Descriptor realizations of* $\mathbf{H}(s,p) = \mathbf{N}(s,p)\mathbf{D}(s,p)^{-1}$ *and* $\mathbf{H}(s,p) = \mathbf{D}(s,p)^{-1}\mathbf{N}(s,p)$ *as in* (4.8) *are given with parameterized versions of* (4.5) *and* (4.6)*, respectively, obtained by replacing the coefficients* $\boldsymbol{\alpha}$ *and* $\boldsymbol{\beta}$ *with* $\tilde{\boldsymbol{\alpha}}(p)$ *and* $\tilde{\boldsymbol{\beta}}(p)$ *from* (4.10b).

(b) *Descriptor realization given with parameterized* (4.5) *is C-controllable and R-observable if and only if* $\mathbf{N}(s,p)$ *and* $\mathbf{D}(s,p)$ *are the right coprime two-variable polynomial matrices, i.e.,* $[\mathbf{N}(s,p)^*, \mathbf{D}(s,p)^*]^*$ *has the full column rank for all finite* $s,p \in \mathbb{C}$ (4.9). *Descriptor realization given with parameterized* (4.6) *is C-observable and R-controllable if and only if* $\mathbf{N}(s,p)$ *and* $\mathbf{D}(s,p)$ *are the left coprime two-variable polynomial matrices, i.e.,* $[\mathbf{N}(s,p), \mathbf{D}(s,p)]$ *has the full row rank for all finite* $s,p \in \mathbb{C}$.

(c) *Descriptor realization given with parameterized* (4.5) *is minimal if and only if* $\mathbf{N}(s,p)$ *and* $\mathbf{D}(s,p)$ *are the right coprime two-variable polynomial matrices and* $\sum_{i=1}^{\overline{n}} \tilde{\boldsymbol{\beta}}_i(p)$ *has the full column rank for all* $p \in \mathbb{C}$. *Descriptor realization given with parameterized* (4.6) *is minimal if and only if* $\mathbf{N}(s,p)$ *and* $\mathbf{D}(s,p)$ *are the left coprime two-variable polynomial matrices and* $\sum_{i=1}^{\overline{n}} \tilde{\boldsymbol{\beta}}_i(p)$ *has the full row rank for all* $p \in \mathbb{C}$.

Remark. Here, just as in the rest of the paper, controllability and observability refer to controllability and observability over $p \in \mathbb{C}$. Again, as in the nonparametric case, we see that introduction of the left polynomial matrix fraction form is necessary to obtain minimal realization.

The proof of Theorem 4.8 is analogous to the proof of Theorem 4.4 given in Appendix C.4. Realizations given with (4.8) are of order $k = \overline{n}n_{min}$ and, when minimal, $k = k_{min} = n + n_{min}$. The latter follows from Lemma 4.6. Therefore, as in the one-variable rational matrix interpolation, the rational matrix cannot always be reconstructed in the minimal descriptor form. However, the presented framework is a useful tool for the low-order parametric system identification and model reduction, where the aforementioned limitation tends to diminish.

4.2.2. Identification of low-order parametric descriptor systems.

Given an arbitrary input set, obtained by sampling a parameter-dependent rational or nonrational matrix, we seek a rational matrix of low-order,

$$\hat{\mathbf{H}}(s,p) = \left(\sum_{i=1}^{\overline{n}}\sum_{j=1}^{\overline{m}} \frac{\mathbf{W}_{ij}\hat{\boldsymbol{\alpha}}_{ij}}{(s-\lambda_i)(p-\pi_j)}\right)\left(\sum_{i=1}^{\overline{n}}\sum_{j=1}^{\overline{m}} \frac{\hat{\boldsymbol{\alpha}}_{ij}}{(s-\lambda_i)(p-\pi_j)}\right)^{-1} \text{for } n_y \geq n_u, \tag{4.14a}$$

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$$\hat{\mathbf{H}}(s,p)=\left(\sum_{i=1}^{\overline{n}}\sum_{j=1}^{\overline{m}}\frac{\hat{\boldsymbol{\alpha}}_{ij}}{(s-\lambda_i)(p-\pi_j)}\right)^{-1}\left(\sum_{i=1}^{\overline{n}}\sum_{j=1}^{\overline{m}}\frac{\hat{\boldsymbol{\alpha}}_{ij}\mathbf{W}_{ij}}{(s-\lambda_i)(p-\pi_j)}\right) \text{ for } n_y\leq n_u, \tag{4.14b}$$

which accurately approximates the sampled data, and its minimal descriptor realization. We determine the desired low degrees $\hat{n}$ and $\hat{m}$ (divisible by n_{min}) of $\hat{\mathbf{H}}(s,p)$ based on the singular value decomposition of the one-variable Loewner matrices from [25]. This is done according to (3.10) where the rank is calculated with some sufficiently small, nonzero tolerance. Then we partition the data such that $\overline{n}=\hat{n}/n_{min}+1$ and $\overline{m}=\hat{m}/n_{min}+1$. With this setting, MIMO $\hat{\mathbb{L}}_2^{r/l}$ is usually numerically full-rank, and coefficients $\hat{\boldsymbol{\alpha}}_{ij}$ are obtained from the n_{min} right singular vectors associated with the n_{min} smallest singular values of $\hat{\mathbb{L}}_2^{r/l}$. Rational approximant $\hat{\mathbf{H}}(s,p)=\hat{\mathbf{N}}(s,p)\hat{\mathbf{D}}(s,p)^{-1}$ or $\hat{\mathbf{H}}(s,p)=\hat{\mathbf{D}}(s,p)^{-1}\hat{\mathbf{N}}(s,p)$ tends to obtain the desired degrees $\hat{n}$ and $\hat{m}$, which are the maximum degrees for chosen $\overline{n}$ and $\overline{m}$, and the coefficients $\hat{\boldsymbol{\alpha}}_{ij}$ tend to be full-rank matrices. Therefore, according to Lemma 4.6, the obtained numerator and denominator polynomial matrices $\hat{\mathbf{N}}(s,p)$ and $\hat{\mathbf{D}}(s,p)$ are coprime, and the descriptor realization is C-controllable and R-observable (C-observable and R-controllable). Furthermore, minimality of realization over $p\in\mathbb{C}$ is achieved for nonsquare transfer matrices, while for square matrices the system is usually minimal over $p\in\mathbb{C}\backslash\{r_p\}$, where r_p denote zeros of determinant of $\sum_{i=1}^{\overline{n}}\tilde{\boldsymbol{\beta}}_i(p)$ (given by (4.10b)). The procedure for low-order state-space identification is summarized in Algorithm 4.1.

Rational matrix $\hat{\mathbf{H}}(s,p)$ as in (4.7) interpolates the sampled set at the Lagrange nodes (λ_i,π_j) by definition, for nonsingular $\hat{\boldsymbol{\alpha}}_{ij}$. The pointwise approximation error at the rest of the nodes is given in Lemma 4.9.

Lemma 4.9. *Let $\hat{\boldsymbol{\Sigma}}=diag(\hat{\sigma}_1,\ldots,\hat{\sigma}_{n_{min}})$ be the matrix of the n_{min} smallest singular values of $\hat{\mathbb{L}}_2^{r/l}$, and let $\hat{\mathbf{X}},\hat{\mathbf{Y}}$ be the associated n_{min} right and left singular vectors, $\hat{\mathbf{Y}}=[\hat{\mathbf{Y}}^*_{\lambda_1},\ldots,\hat{\mathbf{Y}}^*_{\lambda_{\overline{n}}}\mid \hat{\mathbf{Y}}^*_{\pi_1},\ldots,\hat{\mathbf{Y}}^*_{\pi_{\overline{m}}}\mid \mathbf{Y}^*_{2,11},\ldots,\mathbf{Y}^*_{2,\underline{n}\underline{m}}]^*$, $\hat{\mathbf{Y}}^*_{\lambda_i}=[\hat{\mathbf{Y}}^*_{\lambda_i,1},\ldots,\hat{\mathbf{Y}}^*_{\lambda_i,\underline{m}}]^*$, $\hat{\mathbf{Y}}^*_{\pi_j}=[\hat{\mathbf{Y}}^*_{\pi_j,1},\ldots,\hat{\mathbf{Y}}^*_{\pi_j,\underline{n}}]^*$. Then, by setting $\hat{\mathbb{A}}_2^{r/l}=\hat{\mathbf{X}}$, the pointwise approximation error matrix for $n_y\geq n_u$ is given as*

$$\mathbf{E}_{rr}(\lambda_i,\nu_d)=\boldsymbol{\Phi}_{i,\overline{m}+d}-\hat{\mathbf{H}}(\lambda_i,\nu_d)=\hat{\mathbf{Y}}_{\lambda_i,d}\hat{\boldsymbol{\Sigma}}\left(\sum_{j=1}^{\overline{m}}\frac{\hat{\boldsymbol{\alpha}}_{ij}}{\nu_d-\pi_j}\right)^{-1},$$

$$\mathbf{E}_{rr}(\mu_h,\pi_j)=\boldsymbol{\Phi}_{\overline{n}+h,j}-\hat{\mathbf{H}}(\mu_h,\pi_j)=\hat{\mathbf{Y}}_{\pi_j,h}\hat{\boldsymbol{\Sigma}}\left(\sum_{i=1}^{\overline{n}}\frac{\hat{\boldsymbol{\alpha}}_{ij}}{\mu_h-\lambda_i}\right)^{-1},$$

$$\mathbf{E}_{rr}(\mu_h,\nu_d)=\mathbf{V}_{hd}-\hat{\mathbf{H}}(\mu_h,\nu_d)=\hat{\mathbf{Y}}_{2,hd}\hat{\boldsymbol{\Sigma}}\left(\sum_{i=1}^{\overline{n}}\sum_{j=1}^{\overline{m}}\frac{\hat{\boldsymbol{\alpha}}_{ij}}{(\mu_h-\lambda_i)(\nu_d-\pi_j)}\right)^{-1},$$

for $i=1:\overline{n}, h=1:\underline{n}, j=1:\overline{m}, d=1:\underline{m}$.

Lemma 4.9 extends Corollary 4.3 in [17] to the MIMO case. It shows that for rational approximant $\hat{\mathbf{H}}(s,p)$ given by (4.14a), each entry of the pointwise approximation error matrix

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Algorithm 4.1 Identification of low-order parametric MIMO descriptor systems.

$$\textbf{Input}\text{: Sampled transfer matrix } \{s_i, p_j, \mathbf{\Phi}_{ij} \mid s_i \in \mathbb{C}, p_j \in \mathbb{C}, \mathbf{\Phi}_{ij} \in \mathbb{C}^{n_y \times n_u}\}, \tag{4.15}$$

$$i = 1:N, j = 1:M \tag{4.16}$$

$$\textbf{Output}\text{: Parametric descriptor system } \hat{S}(p) = (\hat{\mathbf{E}},\ \hat{\mathbf{A}}(p),\ \hat{\mathbf{B}},\ \hat{\mathbf{C}}(p)) \tag{4.17}$$

for $i = 1:N$ **do**
  Construct one-variable $\mathbb{L}(s_i)$ from [25]
end for
for $j = 1:M$ **do**
  Construct one-variable $\mathbb{L}(p_j)$ from [25]
end for
Determine reduced degrees $\hat{n}, \hat{m}$ (divisible by n_{min}): $\hat{n} \leftarrow \max\ (\text{rank}(\mathbb{L}(p_j), \sigma_{min_n}))$, $j = 1:M$, $\hat{m} \leftarrow \max(\text{rank}(\mathbb{L}(s_i), \sigma_{min_m}))$, $i{=}1:N$, for some small nonzero $\sigma_{min_n}, \sigma_{\min_m}$
Partition the data such that $\overline{n} = \hat{n}/n_{min} + 1$, $\overline{m} = \hat{m}/n_{min} + 1$ for $n_{min}{=}\min(n_y, n_u)$
Construct $\hat{\mathbb{L}}_2^{r/l}$ as in (3.9) with entries given by (4.11)
Calculate singular value decomposition: $[\hat{\mathbf{Y}}_2, \hat{\mathbf{\Sigma}}_2, \hat{\mathbf{X}}_2] = \text{svd}(\hat{\mathbb{L}}_2^{r/l})$;
$\hat{\mathbb{A}}_2^{r/l} \leftarrow \hat{\mathbf{X}}_2(:, end - n_{min} + 1 : end)$;
Calculate coefficients $\tilde{\boldsymbol{\beta}}_i(p)$ and $\tilde{\boldsymbol{\alpha}}_i(p), i = 1:\overline{n}$, as in (4.10b)
Build realization $\hat{S}(p)$ according to (4.5) for $n_y \geq n_u$ and (4.6) for $n_y \leq n_u$ with coefficients $\tilde{\boldsymbol{\beta}}_i(p)$ and $\tilde{\boldsymbol{\alpha}}_i(p)$

is given as a linear combination of the n_{min} smallest singular values of matrix $\hat{\mathbb{L}}_2$. The same holds for $\hat{\mathbf{H}}(s,p)$ given by (4.14b).

In conclusion, the presented methodology allows us to find the low-order two-variable rational approximant of the given sampled data and its minimal realization. The user can choose degrees of the rational approximant by setting appropriate values for $\overline{n}$ and $\overline{m}$ that are suggested by the singular value decomposition of the one-variable Loewner matrices.

5. Stability-preserving postprocessing methods.

Stability of the descriptor systems built with the Loewner frameworks presented in section 4 is not guaranteed. Therefore, to accurately capture the dynamic behavior of a stable original system, a stability-preserving postprocessing method is needed. In this work, we apply the approach developed in [8]. This postprocessing method combines the commonly used sign-pole-flipping approach with the iterative procedure for improving the accuracy after stability enforcement. The method can be applied to a nonparametric descriptor system or to a parametric system which is evaluated for a fixed parameter value. Therefore, we summarize the procedure as given for the nonparametric descriptor realization.

First, matrix pencil $(\hat{\mathbf{A}}, \hat{\mathbf{E}})$ is brought to Schur form, for which the generalized eigenvalues are defined by diagonal (real poles) or block diagonal terms (complex poles). The unstable poles are mirrored by flipping the signs on diagonal elements of matrix $\hat{\mathbf{E}}$ for real poles, and

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flipping the signs of elements on the diagonal which define complex poles of matrix $\hat{\mathbf{A}}$, as suggested in [8]. This yields the matrix pencil $(\hat{\mathbf{A}}_s, \hat{\mathbf{E}}_s)$ with stable poles. However, accuracy of the transfer matrix might be lost. To improve the accuracy of the stable system with mirrored poles, matrices $\hat{\mathbf{B}}$ and $\hat{\mathbf{C}}$ are updated to mitigate the effect of changing matrices $\hat{\mathbf{A}}$ and $\hat{\mathbf{E}}$. This is done iteratively by solving the following least-squares problem:

$$\begin{bmatrix} \hat{\mathbf{C}}_{s,i}(s_1\hat{\mathbf{E}}_s - \hat{\mathbf{A}}_s)^{-1} \\ \vdots \\ \hat{\mathbf{C}}_{s,i}(s_N\hat{\mathbf{E}}_s - \hat{\mathbf{A}}_s)^{-1} \end{bmatrix} \hat{\mathbf{B}}_{s,i+1} = \begin{bmatrix} \hat{\mathbf{H}}(s_1) \\ \vdots \\ \hat{\mathbf{H}}(s_N) \end{bmatrix},$$

$$\hat{\mathbf{C}}_{s,i+1}\left[(s_1\hat{\mathbf{E}}_s - \hat{\mathbf{A}}_s)^{-1}\hat{\mathbf{B}}_{s,i+1} \quad \dots \quad (s_N\hat{\mathbf{E}}_s - \hat{\mathbf{A}}_s)^{-1}\hat{\mathbf{B}}_{s,i+1}\right] = \left[\hat{\mathbf{H}}(s_1) \quad \dots \quad \hat{\mathbf{H}}(s_N)\right],$$

until a specific tolerance or a maximum number of iterations is reached. For the procedure of avoiding complex arithmetic and obtaining improved matrices with real entries, in the case of complex data, we refer the reader to the original work [8].

6. Examples.

In this section we apply the methodologies presented in section 4 to several examples. First, we illustrate the approaches for solving one-variable and two-variable rational matrix interpolation problems in the Loewner framework, and then we apply the developed methodology to low-order parametric state-space identification of an aerodynamic system.

6.1. One-variable rational matrix.

Here we illustrate the approach for solving the one-variable rational matrix interpolation problem from subsubsection 4.1.1 by reconstructing a rectangular rational matrix in the minimal descriptor form from its own samples. We consider both the case when $n_y > n_u$ and the case when $n_y < n_u$.

(a) We consider an adapted version of the rational matrix used in [23], with $n_y > n_u$ and of degree $n = 2$,

$$\mathbf{H}_1(s) = \begin{bmatrix} \frac{1}{1+2s} & \frac{-1}{1+2s} \\ \frac{2+5s}{1+2s} & \frac{3+7s}{1+2s} \\ \frac{1+6s}{1+2s} & \frac{4+9s}{1+2s} \end{bmatrix}.$$

To reconstruct this rational matrix in the minimal realization form, we need to use a rational interpolant in the right polynomial matrix fraction description (4.1a) for interpolation of its samples. The dimension of the associated realization (given by (4.5)) when minimal is $k_{min} = n + n_u = 4$. Therefore, the number of Lagrange nodes is chosen to be $\overline{n} = n/n_u + 1 = 2$. The function is evaluated at four points, $\lambda_1 = 0$, $\lambda_2 = 2$, $\mu_1 = 1$, $\mu_2 = 3$, and the right MIMO Loewner matrix (4.2a) is built as follows:

$$\mathbb{L}^r = \begin{bmatrix} -\frac{2}{3} & \frac{2}{3} & -\frac{2}{5} & \frac{2}{5} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{15} & \frac{1}{15} \\ \frac{4}{3} & \frac{1}{3} & \frac{4}{15} & \frac{1}{15} \\ -\frac{2}{7} & \frac{2}{7} & -\frac{2}{35} & \frac{2}{35} \\ \frac{1}{7} & \frac{1}{7} & \frac{1}{35} & \frac{1}{35} \\ \frac{4}{7} & \frac{1}{7} & \frac{4}{35} & \frac{1}{35} \end{bmatrix}. \tag{6.1}$$

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The rank of $\mathbb{L}^r$ is equal to $n=2$, and therefore its null space, here given in rational basis, is spanned with two vectors,

$$\mathbb{A}^r = \begin{bmatrix} -0.2 & 0 \\ 0 & -0.2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The numerator and denominator coefficients are the following:

$$\begin{aligned} &\boldsymbol{\alpha}_1 = -0.2\mathbf{I}_{2\times 2}, \qquad \boldsymbol{\alpha}_2 = \mathbf{I}_{2\times 2}, \\ &\boldsymbol{\beta}_1 = \begin{bmatrix} -\frac{1}{5} & \frac{1}{5} \\ -\frac{2}{5} & -\frac{3}{5} \\ -\frac{1}{5} & -\frac{4}{5} \end{bmatrix}, \quad \boldsymbol{\beta}_2 = \begin{bmatrix} \frac{1}{5} & -\frac{1}{5} \\ \frac{12}{5} & \frac{17}{5} \\ \frac{13}{5} & \frac{22}{5} \end{bmatrix}. \end{aligned} \tag{6.2}$$

Finally, we reconstruct the rational matrix $\mathbf{H}_1(s)$ in the descriptor form using (4.5),

$$\begin{aligned} &\mathbf{E} = \begin{bmatrix} \mathbf{I}_{2\times 2} & -\mathbf{I}_{2\times 2} \\ \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 2} \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} \mathbf{0}_{2\times 2} & -2\mathbf{I}_{2\times 2} \\ \boldsymbol{\alpha}_1 & \boldsymbol{\alpha}_2 \end{bmatrix}, \\ &\mathbf{C} = \begin{bmatrix} \boldsymbol{\beta}_1 & \boldsymbol{\beta}_2 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} \mathbf{0} \\ \mathbf{I}_{2\times 2} \end{bmatrix}. \end{aligned} \tag{6.3}$$

Constructed realization is minimal since it is of dimension $k_{min} = n + n_u = 4$–, and $\sum_{i=1}^{2} \boldsymbol{\beta}_i$ has full column rank (see Theorem 4.4). We conclude that it is possible to reconstruct the given rational matrix in the minimal descriptor form using the framework from subsubsection 4.1.1.

(b) Consider the transpose of the rectangular rational matrix $\mathbf{H}_1(s)$,

$$\mathbf{H}_2(s) = \begin{bmatrix} \frac{1}{1+2s} & \frac{2+5s}{1+2s} & \frac{1+6s}{1+2s} \\ \frac{-1}{1+2s} & \frac{3+7s}{1+2s} & \frac{4+9s}{1+2s} \end{bmatrix}.$$

Since $n_y < n_u$, we use rational interpolant in the left polynomial matrix fraction description (4.1b) to reconstruct this rational matrix in the minimal realization form with $k_{min} = n + n_y = 4$. For $\mathbf{H}_2(s)$, the left MIMO Loewner matrix is the same as the right MIMO Loewner matrix (6.1) from the previous example, $\mathbb{L}^l(\mathbf{H}_2(s)) = \mathbb{L}^r(\mathbf{H}_1(s))$, and coefficient matrices equal the transpose of $\boldsymbol{\alpha}_i, \boldsymbol{\beta}_i$ as in (6.2). Descriptor realization is then given with (4.6),

$$\begin{aligned} &\mathbf{E} = \begin{bmatrix} \mathbf{I}_{2\times 2} & \mathbf{0}_{2\times 2} \\ -\mathbf{I}_{2\times 2} & \mathbf{0}_{2\times 2} \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} \mathbf{0}_{2\times 2} & \boldsymbol{\alpha}_1 \\ -2\mathbf{I}_{2\times 2} & \boldsymbol{\alpha}_2 \end{bmatrix}, \\ &\mathbf{C} = \begin{bmatrix} \mathbf{0} & \mathbf{I}_{2\times 2} \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} \boldsymbol{\beta}_1 & \boldsymbol{\beta}_2 \end{bmatrix}^*. \end{aligned} \tag{6.4}$$

This realization recovers the original matrix $\mathbf{H}_2(s)$ and is C-controllable and C-observable, i.e., minimal.

6.2. Two-variable rational matrix. Next, we illustrate the approach for solving the two-variable rational matrix interpolation problem from subsubsection 4.2.1 by reconstructing a rectangular two-variable rational matrix in the minimal descriptor form from its own samples.

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Again, as in the nonparametric case, we consider both the case when $n_y > n_u$ and the case when $n_y < n_u$.

(a) We adapt the two-variable rational matrix used in [23] to obtain a rectangular matrix with $n_y > n_u$,

$$\mathbf{H}_1(s,p) = \begin{bmatrix} \frac{s+1}{2s+3p+sp-1} & \frac{s-1}{2s+3p+sp-1} \\ \frac{s+5p+9sp-1}{2s+3p+sp-1} & \frac{3s+7p+11sp-3}{2s+3p+sp-1} \\ \frac{2s+6p+10sp-2}{2s+3p+sp-1} & \frac{4s+8p+12sp-4}{2s+3p+sp-1} \end{bmatrix}.$$

The degrees of $\mathbf{H}_1(s,p)$ in s and p are $n=2$ and $m=2$. We want to reconstruct this two-variable rational matrix in the minimal realization form, and therefore we use the parameterized barycentric formula in the right polynomial matrix fraction description (4.8a). Furthermore, the total number of Lagrange nodes is set to $\overline{n} = n/n_{min}+1 = 2$ and $\overline{m} = m/n_{min}+1 = 2$. The function is evaluated at four complex variable values, $\lambda_1 = 1/2, \lambda_2 = 2, \mu_1 = 3/2, \mu_2 = 3$, and four parameter values, $\pi_1 = -2, \pi_2 = -1$, $\nu_1 = -3/2, \nu_2 = -1/2$. The two-variable right MIMO Loewner matrix $\hat{\mathbb{L}}_2^r$ (defined in (3.9) and (4.12)) of dimension 12×8 is built. The rank of $\hat{\mathbb{L}}_2^r$ is equal to 6, and therefore its null space is spanned with two vectors,

$$\mathbb{A}_2^r = \begin{bmatrix} \frac{7}{2} & 0 & -\frac{7}{4} & 0 & -\frac{7}{2} & 0 & 1 & 0 \\ 0 & \frac{7}{2} & 0 & -\frac{7}{4} & 0 & -\frac{7}{2} & 0 & 1 \end{bmatrix}^*.$$

Using (3.12), parameter-dependent coefficients are obtained,

$$\tilde{\boldsymbol{\alpha}}_1(p) = \frac{7}{4}p\mathbf{I}_{2\times 2}, \qquad \tilde{\boldsymbol{\beta}}_1(p) = \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} \\ \frac{19}{4}p - \frac{1}{4} & \frac{25}{4}p - \frac{3}{4} \\ \frac{11}{2}p - \frac{1}{2} & 7p - 1 \end{bmatrix},$$
$$\tilde{\boldsymbol{\alpha}}_2(p) = \left(-\frac{5}{2}p - \frac{3}{2}\right)\mathbf{I}_{2\times 2}, \quad \tilde{\boldsymbol{\beta}}_2(p) = \begin{bmatrix} -\frac{3}{2} & -\frac{1}{2} \\ -\frac{23}{2}p - \frac{1}{2} & -\frac{29}{2}p - \frac{3}{2} \\ -13p - 1 & -16p - 2 \end{bmatrix}. \tag{6.5}$$

Finally, descriptor realization which recovers the original two-variable matrix is obtained from (4.5) by replacing coefficients $\boldsymbol{\alpha}_i$ and $\boldsymbol{\beta}_i$ with $\tilde{\boldsymbol{\alpha}}_i(p)$ and $\tilde{\boldsymbol{\beta}}_i(p)$,

$$\mathbf{E} = \begin{bmatrix} \mathbf{I}_{2\times 2} & -\mathbf{I}_{2\times 2} \\ \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 2} \end{bmatrix}, \quad \mathbf{A}(p) = \begin{bmatrix} 0.5\mathbf{I}_{2\times 2} & -2\mathbf{I}_{2\times 2} \\ -\tilde{\boldsymbol{\alpha}}_1(p) & -\tilde{\boldsymbol{\alpha}}_2(p) \end{bmatrix},$$
$$\mathbf{C}(p) = \begin{bmatrix} \tilde{\boldsymbol{\beta}}_1(p) & \tilde{\boldsymbol{\beta}}_2(p) \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} \mathbf{0} \\ \mathbf{I}_{2\times 2} \end{bmatrix}.$$

This realization meets the requirements for minimality over $p \in \mathbb{C}$ stated in Theorem 4.8.

(b) Consider the transpose of the rectangular rational matrix $\mathbf{H}_1(s,p)$,

$$\mathbf{H}_2(s,p) = \begin{bmatrix} \frac{s+1}{2s+3p+sp-1} & \frac{s+5p+9sp-1}{2s+3p+sp-1} & \frac{2s+6p+10sp-2}{2s+3p+sp-1} \\ \frac{s-1}{2s+3p+sp-1} & \frac{3s+7p+11sp-3}{2s+3p+sp-1} & \frac{4s+8p+12sp-4}{2s+3p+sp-1} \end{bmatrix}.$$

We reconstruct this two-variable rational matrix with $n_y < n_u$ using the parameterized barycentric formula in the left polynomial matrix fraction description (4.8b). Since $\mathbf{H}_2(s,p) =$

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$\mathbf{H}_1^*(s,p)$, it follows that $\mathbb{L}_2^l(\mathbf{H}_2(s,p)) = \mathbb{L}_2^r(\mathbf{H}_1(s,p))$, and thus the polynomial coefficient matrices are the transpose of $\tilde{\boldsymbol{\alpha}}_i(p)$ and $\tilde{\boldsymbol{\beta}}_i(p)$ from the previous example. The descriptor system can be obtained from (4.6) by replacing coefficients $\boldsymbol{\alpha}_i$ and $\boldsymbol{\beta}_i$ with $\tilde{\boldsymbol{\alpha}}_i(p)$ and $\tilde{\boldsymbol{\beta}}_i^*(p)$ from (6.5),

$$\mathbf{E} = \begin{bmatrix} \mathbf{I}_{2\times 2} & \mathbf{0}_{2\times 2} \\ -\mathbf{I}_{2\times 2} & \mathbf{0}_{2\times 2} \end{bmatrix}, \quad \mathbf{A}(p) = \begin{bmatrix} 0.5\mathbf{I}_{2\times 2} & -\tilde{\boldsymbol{\alpha}}_1(p) \\ -2\mathbf{I}_{2\times 2} & -\tilde{\boldsymbol{\alpha}}_2(p) \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} \mathbf{0} & \mathbf{I}_{2\times 2} \end{bmatrix}, \qquad \mathbf{B}(p) = \begin{bmatrix} \tilde{\boldsymbol{\beta}}_1^*(p) \\ \tilde{\boldsymbol{\beta}}_2^*(p) \end{bmatrix}.$$

This realization is minimal over $p \in \mathbb{C}$.

We conclude that the developed approach for two-variable rational matrix interpolation is able to reconstruct the given two-variable rational matrices from its samples in the minimal descriptor form.

6.3. Aerodynamic system. Here we apply the developed parametric Loewner framework to low-order parametric state-space identification of an aerodynamic system. The considered aerodynamic system models the flow around the aircraft wing section (modeled as a flat plate) with three degrees of freedom, namely, heave h, pitch θ, and flap deflection δ. The system is described with 2×3 transfer matrix $\mathbf{H}_a(s,p)$,

$$\mathbf{H}_a(s,p) = \begin{bmatrix} C_{l_h} & C_{l_\theta} & C_{l_\delta} \\ C_{m_h} & C_{m_\theta} & C_{m_\delta} \end{bmatrix}.$$

The first row and the second row contain derivatives of lift coefficient C_l and moment coefficient C_m with respect to h, θ–, and δ. The considered parameter is Mach number which is a real number, $p \in \mathbb{R}$. Due to the nature of aerodynamic loads, $\mathbf{H}_a(s,p)$ is not a rational matrix. Furthermore, the considered aerodynamic system is asymptotically stable. As stated in section 2, our goal is to find the parametric descriptor system of low order which closely approximates $\mathbf{H}_a(s,p)$ and accurately captures dynamic behavior of the original system.

First, the aerodynamic transfer matrix is sampled, $\boldsymbol{\Phi}_{ij} = \mathbf{H}_a(s_i,p_j)$, for $N = 40$ logarithmically spaced frequency values on the interval $s = i\omega \in [4 \cdot 10^{-6}\ 600]$ and $M = 30$ parameter values, $p \in [0.1\ 0.7]$. Sampling of the transfer matrix in p is logarithmic with higher density of sampling points at the edges of the observed interval. This is done to avoid the Runge effect. To ensure the real system, we add complex conjugate pairs of data for each frequency (except the first one which is approximately zero) which results in a total of $2N - 1 = 79$ sampling points in s. The second step is to find the appropriate degrees of rational approximant $\hat{\mathbf{H}}(s,p)$ in s and p such that it accurately approximates the sampled set. This is done by constructing one-variable Loewner matrices $\mathbb{L}(p_j)$ and $\mathbb{L}(s_i)$ from [25] for all s and p.

Figure 1 shows that all $\mathbb{L}(p_j)$ and $\mathbb{L}(s_i)$ have full numerical ranks, which is characteristic for nonrational matrix samples. Desired degrees are then chosen as $\hat{n}$ =max (rank($\mathbb{L}(p_j), \sigma_{min_n}$)), $j = 1 : M$, and $\hat{m}$ =max(rank($\mathbb{L}(s_i), \sigma_{min_m}$)), $i = 1 : N$, such that the tolerances σ_{min_n} and σ_{min_m} are sufficiently small and the degrees are divisible by n_{min}. In order to analyze the influence of the degree on approximation error, we build two descriptor systems, $\hat{S}_1(p)$ and $\hat{S}_2(p)$, with the associated rational matrices $\hat{\mathbf{H}}_1(s,p)$ and $\hat{\mathbf{H}}_2(s,p)$. We choose $(\hat{n}_1,\ \hat{m}_1) =$

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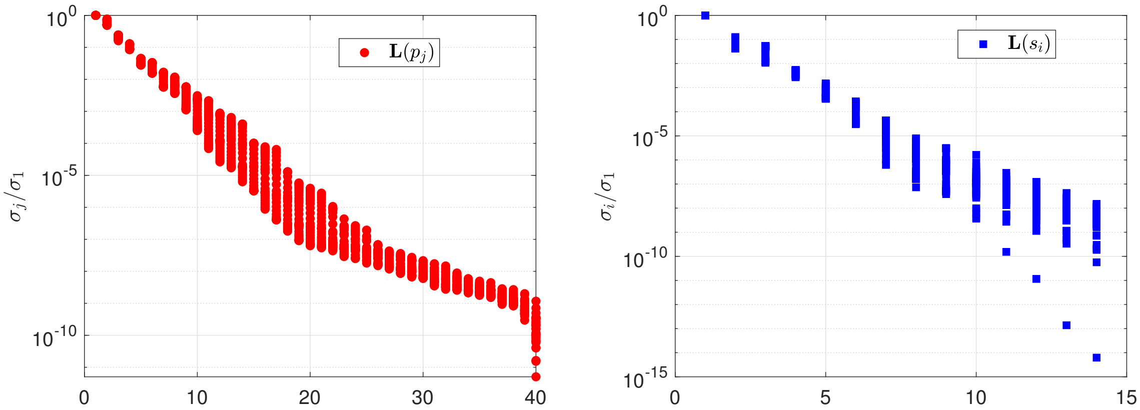

(a) $\mathbb{L}(p_j), j = 1:M$ (b) $\mathbb{L}(s_i), i = 1:N$

Figure 1. *Singular values of one-variable Loewner matrices $\mathbb{L}(p_j)$ and $\mathbb{L}(s_i)$ for $j = 1:M$, $i = 1:N$.*

$(36, 14)$ for $\hat{\mathbf{H}}_1(s,p)$ and $(\hat{n}_2,\ \hat{m}_2) = (20, 10)$ for $\hat{\mathbf{H}}_1(s,p)$ (which correspond to sufficiently small singular value decomposition tolerances as visible from Figure 1). Corresponding orders are $k_1 = 38$ and $k_2 = 22$. Next, we set $\overline{n}_1 = \hat{n}_1/2 + 1 = 19$, $\overline{m}_1 = \hat{m}_1/2 + 1 = 8$ and $\overline{n}_2 = \hat{n}_2/2 + 1 = 11$, $\overline{m}_2 = \hat{m}_2/2 + 1 = 6$. Since $n_y < n_u$, we build the left (MIMO) Loewner matrix $\hat{\mathbb{L}}_2^l$ given by (4.13) and obtain coefficients $\hat{\boldsymbol{\alpha}}_{ij}$ from the right singular vectors associated to its two smallest singular values. Indeed, the built rational matrices obtain maximum degrees $\hat{n}$ and $\hat{m}$ for given $\overline{n}$ and $\overline{m}$, as expected. This can be checked by generating a sufficient amount of samples of $\hat{\mathbf{H}}_1(s,p)$ and $\hat{\mathbf{H}}_2(s,p)$ and calculating the ranks of the associated one-variable Loewner matrices. Furthermore, $\hat{\boldsymbol{\alpha}}_{ij}$ are full-rank matrices, and the sum of obtained matrices $\tilde{\boldsymbol{\beta}}_i(p)$ is of full row rank for all $p \in \mathbb{R}$. Therefore, both descriptor systems are minimal over $p \in \mathbb{R}$ (this follows from Lemma 4.6 and Theorem 4.8).

Next, we analyze the approximation errors of the two descriptor systems in the frequency domain. For this purpose, the aerodynamic transfer matrix is additionally calculated on a finer grid which consist of 100 linearly spaced parameter values on the observed range and the sample points in s. To illustrate the accuracy of the two descriptor systems, we show the absolute values of the transfer matrix entries $(1,2)$ and $(2,2)$ for two parameter values, $p = 0.16$ and $p = 0.6$, in Figures 2a and 2b. The first parameter value is contained in the sampled set and is partitioned as the Lagrange node π. The second parameter value is contained in the finer set and not in the sampled set. We can see that for $p = 0.16$, both systems have approximation errors equal to (numerical) zero at $s = \lambda_i$, which holds by construction. For both parameters, $\hat{\mathbf{H}}_1(s,p)$ approximates the data more accurately. Additionally, we check the accuracy of the models on the whole observed interval by calculating the normalized $\mathcal{H}_2$ norm (defined in [22]) of the pointwise error matrix $\mathbf{E}_{rr}(s,p)$. The error matrix is calculated on the fine grid and shown in Figure 2c. Again, as expected, $\hat{S}_1(p)$ shows smaller approximation errors for all p. This is consistent with Lemma 4.9 since the smallest singular values of the associated $\hat{\mathbb{L}}_2$ for $(\hat{n}_2,\ \hat{m}_2) = (36, 14)$ are $(\hat{\sigma}_1 = 9.3 \cdot 10^{-5}, \hat{\sigma}_2 = 8.3 \cdot 10^{-5})$, while for $(\hat{n}_2,\ \hat{m}_2) = (20, 10)$

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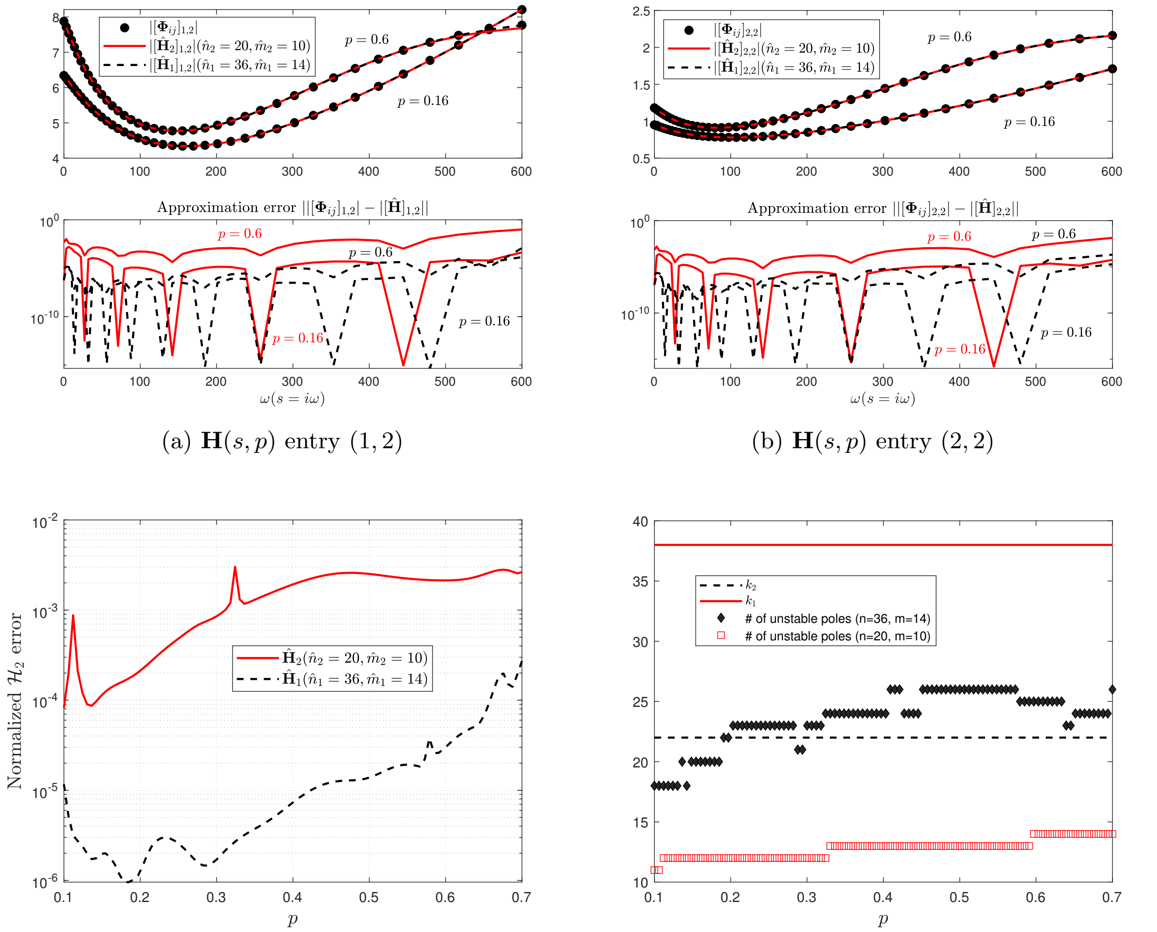

(a) $\mathbf{H}(s,p)$ entry $(1,2)$

(b) $\mathbf{H}(s,p)$ entry $(2,2)$

(c) Normalized $\mathcal{H}_2$ norm of pointwise error matrix.

(d) Dimensions and number of unstable poles.

Figure 2. *Descriptor systems $\hat{S}_1(p)$ with degrees $(\hat{n}_1, \hat{m}_1) = (38,14)$ and $\hat{S}_2(p)$ with degrees $(\hat{n}_2, \hat{m}_2)=(20,10)$.*

the smallest singular values are $(\hat{\sigma}_1 = 9.8 \cdot 10^{-4}, \hat{\sigma}_2 = 8 \cdot 10^{-4})$. Moreover, we can see that the approximation error is increasing with increasing parameter value, as the transfer matrix becomes more curvy.

In addition to the transfer matrix, the modeled system also needs to closely match the dynamic behavior of the original system. Thus, the poles of the system need to be constrained to the left half of the complex plane since the original system is stable. However, the Loewner framework does not guarantee preservation of the stability. This is shown in Figure 2d, which shows the number of unstable poles of $\hat{S}_1(p)$ and $\hat{S}_2(p)$. For both systems, this number is significant. Therefore we apply the postprocessing stability-preserving technique explained in section 5. For the sake of brevity, we continue our analysis only for $\hat{S}_2(p)$. The accuracy of $\hat{\mathbf{H}}_s(s,p)$ (of degrees $\hat{n}_2$ and $\hat{m}_2$) associated with the stable system $\hat{S}_s$ is illustrated in Figures 3a and 3b. It can be seen that the accuracy is still high. This holds for the whole observed interval of parameter p, as shown in Figure 3c.

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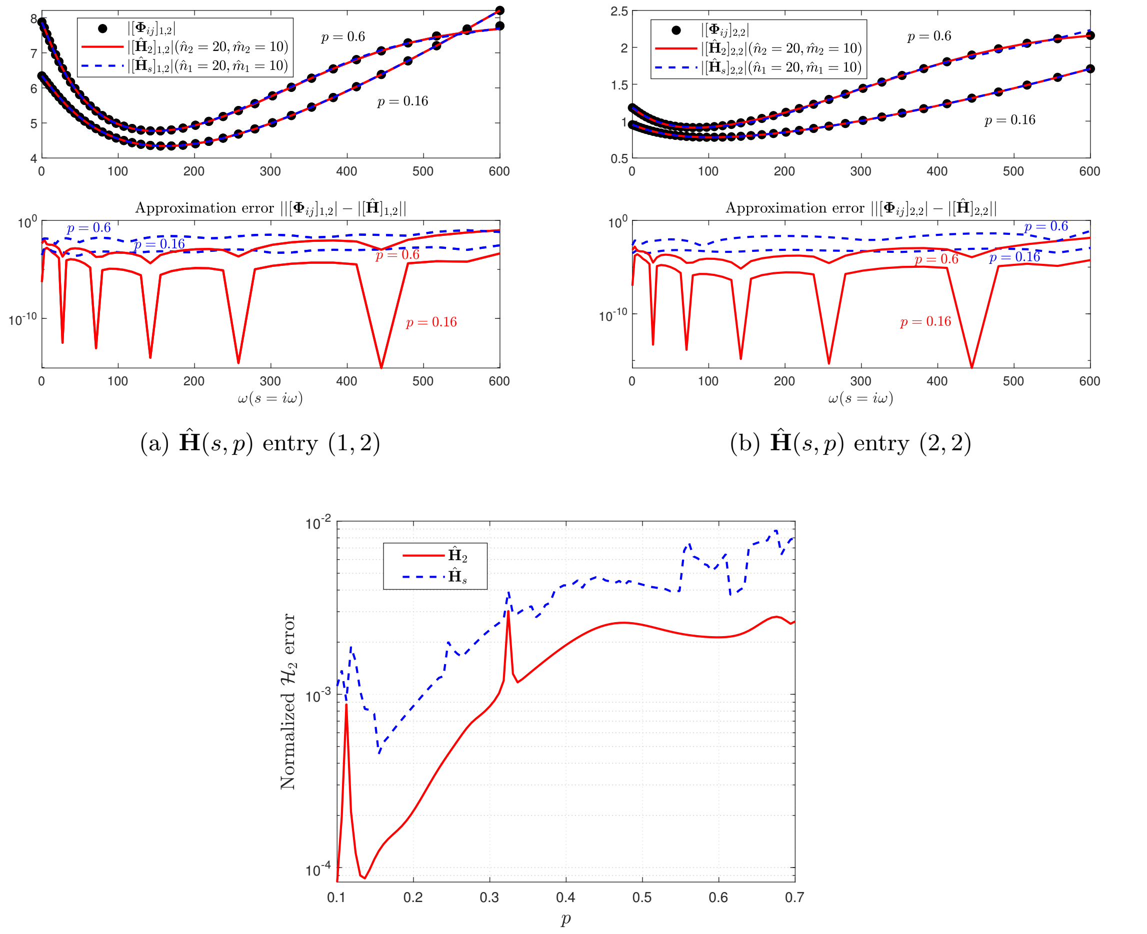

(a) $\hat{\mathbf{H}}(s,p)$ entry $(1,2)$

(b) $\hat{\mathbf{H}}(s,p)$ entry $(2,2)$

(c) Normalized $\mathcal{H}_2$ norm of pointwise error matrix

Figure 3. *Unstable descriptor system $\hat{S}_2(p)$ and stable descriptor system $\hat{S}_s$ with degrees $(\hat{n},\ \hat{m}) = (20{,}10)$.*

Finally, we can evaluate the dynamic behavior of the built aerodynamic model. For this purpose, the system is excited with a pulse function given as

$$\mathbf{u}(t) = \begin{bmatrix} 4u_{h0}(t/t_0)^2 e^{(2-1/(1-t/t_0))} \\ 4u_{\theta 0}(t/t_0)^2 e^{(2-1/(1-t/t_0))} \\ 4u_{\delta 0}(t/t_0)^2 e^{(2-1/(1-t/t_0))} \end{bmatrix}, \quad 0 \le t \le t_0,$$

$$\mathbf{u}(t) = \mathbf{0}, \qquad t > t_0,$$

where $t_0 = 0.5s$ and $u_{h0} = 0.01$, $u_{\theta 0} = u_{\delta 0} = \pi/180$. The input is band-limited, $\omega \in [0\ 400]$ rad/s. State-space equations are solved for time $t \in [0\ 0.1]s$ using the differential algebraic equation solver [30] based on the algorithms described in [29]. The reference solution is calculated in the frequency domain and brought back to the time domain by means of the fast Fourier transform. In order to avoid introducing the interpolation error in the frequency domain solution, a new set of aerodynamic transfer matrices is calculated at the reduced

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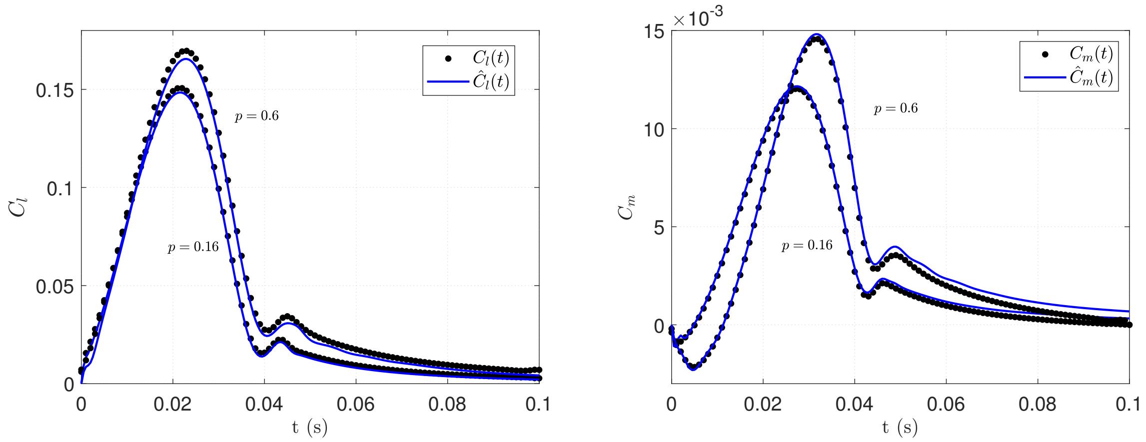

(a) C_l (b) C_m

Figure 4. *Dynamic responses, C_l and C_m, to pulse excitation predicted with stable descriptor system.*

frequencies which are defined with the prescribed time. Dynamic responses C_l and C_m are shown for the same two parameters as before, $p = 0.6$ and $p = 0.16$, in Figure 4. Consistent with the other results, the accuracy of the dynamic responses is higher for $p = 0.16$. We can conclude that, in addition to good accuracy in the frequency domain, built models accurately capture the dynamic behavior of the original system in the time domain as well.

7. Conclusion. We have introduced a novel data-driven method for identification of parametric MIMO descriptor systems of low order from transfer matrix samples. The method is based on the two-variable Lagrange rational matrix interpolation within the Loewner framework. The novelty of the presented method lies in the fact that by introducing the new barycentric formula in the left and right polynomial matrix fraction forms, construction of completely controllable and observable parametric descriptor systems is enabled. Reduction of both the order of the system and complexity of its parameter dependence is done by choosing low degrees of rational approximant. Appropriate degrees which yield small approximation errors are suggested by the singular value decomposition of the one-variable Loewner matrices. First, the proposed approaches for solving the one-variable and two-variable rational matrix interpolation problems are illustrated on academic examples. In these examples, rational matrices are reconstructed from their samples in the minimal descriptor realization form. The proposed method is then applied to parametric state-space modeling of aerodynamic loads, proving that it is capable of identifying minimal descriptor systems of low order and high accuracy.

Appendix A. Avoiding complex arithmetic. Often in real applications, the variable s is complex while the parameter p is real. This results in the complex-valued Loewner matrices and realizations with complex-valued matrices. However, in practice, we want to avoid this.

The procedure for avoiding complex arithmetic and obtaining realization with real-valued matrices for nonparametric and parametric SISO cases is shown in [16, 17]. Here, we generalize it to handle MIMO systems.

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A.1. Nonparametric case. Our first step is to augment the complex input data by including its complex conjugate pairs. Here we assume that all s_i are complex except for $s_1 = \lambda_1$, as in [17]. The data is then partitioned as

$$[s_1, \overline{s}_1, \ldots, s_N, \overline{s}_N] = \left[\lambda_1, \lambda_2, \overline{\lambda}_2, \ldots, \lambda_{\overline{n}_r}, \overline{\lambda}_{\overline{n}_r}\right] \cup \left[\mu_1, \overline{\mu}_1, \ldots, \mu_{\overline{n}_l}, \overline{\mu}_{\overline{n}_l}\right],$$

where the number of right and left samples is $\overline{n} = 2\overline{n}_r - 1$ and $\underline{n} = 2\overline{n}_l$, and the total number of samples is $\overline{n} + \underline{n} = 2N - 1$. Samples $\mathbf{\Phi}$ are extended and partitioned in the same fashion. To obtain real coefficients in the barycentric formula, we need to set $\mathbf{\Phi}(\overline{s}_i) = \overline{\Phi(s_i)}$ [17]. The resulting Loewner matrix $\mathbb{L}$ as in (3.4) and (4.2) has complex conjugate structure. The real-valued Loewner matrix is obtained by the unitary transformation

$$\mathbb{L}_R = \mathbf{Y}\mathbb{L}\mathbf{X}, \tag{A.1}$$

where

$$\mathbf{Y} = \mathrm{diag}(\mathbf{I}_{\overline{n}_l} \otimes \mathbf{J}) \otimes \mathbf{I}_{n_{max}}, \quad \mathbf{X} = \mathrm{diag}(1, \mathbf{I}_{\overline{n}_r - 1} \otimes \mathbf{J}^*) \otimes \mathbf{I}_{n_{min}}, \tag{A.2}$$

$\mathbf{J}$ is a unitary matrix, $\mathbf{J} = \frac{1}{\sqrt{2}}\left[\begin{smallmatrix} 1 & 1 \\ -\imath & \imath \end{smallmatrix}\right]$, and $n_{max} = \max(n_y, n_u)$. This holds for the SISO Loewner matrix ($n_{max} = n_{min} = 1$) (3.4) as well as the right and left MIMO Loewner matrices (4.2a), (4.2b). Null vectors of $\mathbb{L}_R$, denoted as $\mathbb{A}_R$, have only real entries and are given by $\mathbb{A}_R = \mathbf{X}^*\mathbb{A}$. Thus, to avoid complex arithmetic, we calculate singular value decomposition of $\mathbb{L}_R$ and obtain $\mathbb{A}$ (denoted as $\mathbf{a}$ as in (3.4) in the SISO case) through transformation. Once $\mathbb{A}$ is found, we can construct realization $(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C})$ with (3.5), (4.5), (4.6), which has complex conjugate structure as well. Finally, real-valued state-space matrices are obtained with the transformation

$$\begin{aligned} &\mathbf{E}_R = \mathbf{W}\mathbf{E}\mathbf{V}^*,\ \mathbf{A}_R = \mathbf{W}\mathbf{A}\mathbf{V}^*,\ \mathbf{B}_R = \mathbf{W}\mathbf{B},\ \mathbf{C}_R = \mathbf{C}\mathbf{V}^* \text{ for } n_y \geq n_u, \\ &\mathbf{E}_R = \mathbf{V}\mathbf{E}\mathbf{W}^*,\ \mathbf{A}_R = \mathbf{V}\mathbf{A}\mathbf{W}^*,\ \mathbf{B}_R = \mathbf{V}\mathbf{B},\ \mathbf{C}_R = \mathbf{C}\mathbf{W}^* \text{ for } n_y \leq n_u, \end{aligned} \tag{A.3}$$

where $\mathbf{W} = \mathrm{diag}(\sqrt{2}\mathbf{I}_{\overline{n}_r - 1} \otimes \mathbf{J}, 1) \otimes \mathbf{I}_{n_{min}}$ and $\mathbf{V} = \mathrm{diag}(1, \sqrt{2}\mathbf{I}_{\overline{n}_r - 1} \otimes \mathbf{J}) \otimes \mathbf{I}_{n_{min}}$.

A.2. Parametric case. The procedure for avoiding complex arithmetic in the parametric case, when $p \in \mathbb{R}$, is the same as that for the nonparametric case. Transformation matrices used in the transformation of the two-variable Loewner matrix are given as

$$\mathbf{Y} = \mathrm{diag}(\mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y}_3) \otimes \mathbf{I}_{n_{min}},\ \mathbf{X} = \mathrm{diag}(\mathbf{I}_{\overline{m}}, \mathbf{I}_{\overline{n}_r - 1} \otimes (\mathbf{J}^* \otimes \mathbf{I}_{\overline{m}})) \otimes \mathbf{I}_{n_{min}},$$

where $\mathbf{Y}_1$, $\mathbf{Y}_2$, $\mathbf{Y}_3$ [17] are

$$\begin{aligned} &\mathbf{Y}_1 = \mathrm{diag}(\mathbf{I}_{M - \overline{m}}, \mathbf{I}_{\overline{n}_r - 1} \otimes (\mathbf{J} \otimes \mathbf{I}_{M - \overline{m}})),\ \mathbf{Y}_2 = \mathbf{I}_{(N - \overline{n} + 1)\overline{m}} \otimes \mathbf{J}, \\ &\mathbf{Y}_3 = \mathbf{I}_{N - \overline{n} + 1} \otimes [\mathbf{I}_{M - \overline{m}} \otimes \mathbf{J}(:, 1), \mathbf{I}_{M - \overline{m}} \otimes \mathbf{J}(:, 2)]. \end{aligned}$$

The real-valued two-variable Loewner matrices, $\hat{\mathbb{L}}_{2_R}$ and $\hat{\mathbb{L}}_2$, are then related to each other according to (A.1). The transformation of realization matrices is the same as that given by (A.3).

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Appendix B. Error bound. If the sampled data originates from a known, continuously differentiable (in s), matrix function $\mathbf{R}(s,p)$, we can obtain the error bound over a desired interval $\mathcal{I}$ of complex variable s rather than just at the sample points. Here we give the error bound in Frobenius norm,

$$\|\mathbf{R}(s,p)-\hat{\mathbf{H}}(s,p)\|_F \leq \max_{s\in\mathcal{I}} \left\|\frac{d}{ds}\mathbf{R}(s,p)\right\|_F \cdot \sum_{i=1}^{\bar{n}} \|\tilde{\boldsymbol{\alpha}}_i(p)\|_F \left\| \left(\sum_{i=1}^{\bar{n}} \frac{\tilde{\boldsymbol{\alpha}}_i(p)}{s-\lambda_i} \right)^{-1} \right\|_F ,$$

for a fixed p. This expression generalizes the one given in [17, 20] for the SISO case.

Appendix C. Proofs.

C.1. Proof of Theorem 3.2(b).

Conditions for C-controllability,

$$\operatorname{rank}\left[s\mathbf{E}-\mathbf{A},\mathbf{B}\right] = n+1 \text{ for all finite } s\in\mathbb{C} \text{ and } \operatorname{rank}\left[\mathbf{E},\mathbf{B}\right] = n+1,$$

hold by construction. Thus, realization is always C-controllable. For it to be R-observable, the following must hold:

$$\operatorname{rank}\left[(s\mathbf{E}-\mathbf{A})^*,\mathbf{C}^*\right]^* = n+1 \text{ for all finite } s\in\mathbb{C}. \tag{C.1}$$

If (C.1) fails, there exists a nontrivial null vector $\mathbf{x} = [x_1, x_2, \ldots, x_{n+1}]^*$ such that

$$\begin{bmatrix} s-\lambda_1 & \lambda_2 - s & & \\ \vdots & & \ddots & \\ s-\lambda_1 & & & \lambda_{n+1}-s \\ \alpha_1 & \alpha_2 & \dots & \alpha_{n+1} \\ \beta_1 & \beta_2 & \dots & \beta_{n+1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n+1} \end{bmatrix} = \mathbf{0} \text{ for some finite } s\in\mathbb{C}. \tag{C.2}$$

The first n equations (rows) in (C.2) give $x_i = \frac{s-\lambda_1}{s-\lambda_i}x_1$ for $i = 2:n+1$. Therefore, for a nontrivial $\mathbf{x}$, $x_1 \neq 0$. The last two equations, after multiplying with $l_{s_1}(s)$ and taking into account that $x_1 \neq 0$, yield

$$d(s) = \sum_{i=1}^{n+1} \alpha_i l_{s_i}(s) = 0 \text{ and } n(s) = \sum_{i=1}^{n+1} \beta_i l_{s_i}(s) = 0 \text{ for some finite } s\in\mathbb{C}.$$

This implies that the numerator and denominator polynomials, $n(s)$ and $d(s)$, have a common root, i.e., they are reducible. However, for an $H(s)$ of degree n given with (3.2), this cannot hold. Therefore, the realization is C-controllable and R-observable.

For the realization to be C-observable, (C.1) and

$$\operatorname{rank}\left[\mathbf{E}^*,\mathbf{B}^*\right]^* = n+1 \tag{C.3}$$

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need to hold. If (C.3) does not hold, there exists a null vector $\mathbf{y} = [y, y, \ldots, y]^*$ such that

$$\begin{bmatrix} 1 & -1 & & \\ \vdots & & \ddots & \\ 1 & & & -1 \\ 0 & 0 & \ldots & 0 \\ \beta_1 & \beta_2 & \ldots & \beta_{n+1} \end{bmatrix} \begin{bmatrix} y \\ y \\ \vdots \\ y \end{bmatrix} = \mathbf{0} \text{ for some finite } s \in \mathbb{C}.$$

This implies that the highest numerator coefficient equals zero, $\sum_{i=1}^{n+1} \beta_i = 0$. This holds only for strictly proper $H(s)$. Therefore, this realization is minimal for proper and improper $H(s)$, while for strictly proper $H(s)$ it is C-controllable and R-observable. ■

C.2. Proof of Theorem 3.3(b). Analogous to the nonparametric case, conditions for C-controllability (2.3) and (2.4) hold by construction. Furthermore, R-observability condition (2.5) holds for $H(s,p)$ of degrees (n,m) due to irreducibility of numerator $n(s,p)$ and denominator $d(s,p)$. It can be shown (similar to subsection C.1) that (2.6) holds only if $\sum_{i=1}^{n+1} \tilde{\beta}_i(p) \neq 0$ for every $p \in \mathbb{C}$. This is only true for $H(s,p)$ that is proper or improper in s for every $p \in \mathbb{C}$. Therefore, the realization is C-controllable and C-observable (minimal) if $H(s,p)$ is proper or improper in s for every $p \in \mathbb{C}$; otherwise, it is C-controllable and R-observable. ■

C.3. Proof of Lemma 4.2. If (4.4a) fails, there exists a nonzero left null vector of the MIMO generalized controllability matrix, $\mathbf{x}^*$, and a right null vector of the MIMO generalized observability matrix, $\mathbf{y}$, such that

$$\mathbf{x}^* \left[(\lambda_1 \mathbf{E} - \mathbf{A})^{-1}\mathbf{B} \quad \ldots \quad (\lambda_{\overline{n}} \mathbf{E} - \mathbf{A})^{-1}\mathbf{B} \right] = \mathbf{0}, \tag{C.4a}$$

$$\begin{bmatrix} \mathbf{C}(\mu_1 \mathbf{E} - \mathbf{A})^{-1} \\ \vdots \\ \mathbf{C}(\mu_{\underline{n}} \mathbf{E} - \mathbf{A})^{-1} \end{bmatrix} \mathbf{y} = \mathbf{0}. \tag{C.4b}$$

From (C.4a) it follows that a row vector of transfer matrix $\mathbf{x}^*(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}$ has $\overline{n}$ zeros at $s = \lambda_i, i = 1 : \overline{n}$. Therefore, each entry of transfer matrix $\mathbf{x}^*(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}e_j$, where e_j is a unit vector, has $\overline{n}$ zeros. However, the numerator of $\mathbf{x}^*(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}e_j$ has a degree less than or equal to n, and thus when $\overline{n} > n$, it must be identically zero, $\mathbf{x}^*(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B} = 0$. This violates the requirement that $[\mathbf{E}, \mathbf{A}, \mathbf{B}]$ must be controllable. Similarly, from (C.4b) is follows that when $\underline{n} > n$, numerator $e_i^* \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{y}$ must be identically zero. This contradicts the requirement that $[\mathbf{E}, \mathbf{A}, \mathbf{C}]$ must be observable. The proof for (4.4b) is analogous. ■

C.4. Proof of Theorem 4.4.

(a) First, we denote the matrix pencil $(\mathbf{A}, \mathbf{E})$ given with (4.5) as $\mathbf{J}(s)$ and partition it in the 2×2 block form as

$$\mathbf{J}(s) = \left[\begin{array}{c|cccc} (s - \lambda_1)\mathbf{I}_{n_u} & (\lambda_2 - s)\mathbf{I}_{n_u} & & \\ \vdots & & \ddots & \\ (s - \lambda_1)\mathbf{I}_{n_u} & & & (\lambda_{\overline{n}} - s)\mathbf{I}_{n_u} \\ \hline \boldsymbol{\alpha}_1 & \boldsymbol{\alpha}_2 & \ldots & \boldsymbol{\alpha}_{\overline{n}} \end{array} \right] = \left[\begin{array}{c|c} \mathbf{J}_{11} & \mathbf{J}_{12} \\ \mathbf{J}_{21} & \mathbf{J}_{22} \end{array} \right].$$

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After this partition, we can calculate the inverse of $\mathbf{J}(s)$. Since the inverse of $\mathbf{J}(s)$ is multiplied by $\mathbf{B}$, we are interested only in the last column of $\mathbf{J}^{-1}(s)$,

$$\mathbf{J}^{-1}(s)\mathbf{B} = \begin{bmatrix} (\mathbf{J}_{21} - \mathbf{J}_{22}\mathbf{J}_{12}^{-1}\mathbf{J}_{11})^{-1} \\ -\mathbf{J}_{12}^{-1}\mathbf{J}_{11}(\mathbf{J}_{21} - \mathbf{J}_{22}\mathbf{J}_{12}^{-1}\mathbf{J}_{11})^{-1} \end{bmatrix} = \begin{bmatrix} l_{s_1}(\sum_{i=1}^{\overline{n}} l_i \boldsymbol{\alpha}_i)^{-1} \\ l_{s_2}(\sum_{i=1}^{\overline{n}} l_i \boldsymbol{\alpha}_i)^{-1} \\ \vdots \\ l_{s_{\overline{n}}}(\sum_{i=1}^{\overline{n}} l_i \boldsymbol{\alpha}_i)^{-1} \end{bmatrix}.$$

Finally,

$$\mathbf{C}\mathbf{J}^{-1}(s)\mathbf{B} = \left(\sum_{i=1}^{\overline{n}} l_{s_i}(s)\boldsymbol{\beta}_i\right)\left(\sum_{i=1}^{\overline{n}} l_{s_i}(s)\boldsymbol{\alpha}_i\right)^{-1} = \left(\sum_{i=1}^{\overline{n}} \frac{\boldsymbol{\beta}_i}{s-\lambda_i}\right)\left(\sum_{i=1}^{\overline{n}} \frac{\boldsymbol{\alpha}_i}{s-\lambda_i}\right)^{-1}$$

The proof for (4.6) is analogous.

(b) Realization (4.5) satisfies nonparametric versions of the controllability conditions (2.3) and (2.4) by construction (for distinct λ_i). However, the R-observability condition (2.5) requires $[\mathbf{N}(s)^*, \mathbf{D}(s)^*]^*$ to have full column rank, i.e., requires $\mathbf{N}(s)$ and $\mathbf{D}(s)$ to be right coprime. Realization (4.6) is C-observable by construction. Nonparametric R-controllability condition (2.3) requires $[\mathbf{N}(s), \mathbf{D}(s)]$ to have full row rank, i.e., requires $\mathbf{N}(s)$ and $\mathbf{D}(s)$ to be left coprime.

(c) The descriptor realization given by (4.5) is of dimension $k = \overline{n} \cdot n_u$. In order for it to be minimal, $\mathbf{N}(s)$ and $\mathbf{D}(s)$ need to right coprime, and condition (2.6) needs to hold. The latter requires $\sum_{i=1}^{\overline{n}} \boldsymbol{\beta}_i$ to have full column rank. If the two conditions are fulfilled, the conditions stated in Lemma 4.1 are automatically fulfilled. Therefore $n = (\overline{n}-1)n_u$ and $k_{min} = n + n_u$. The proof for (4.6) is analogous. ■

C.5. Proof of Theorem 4.7. First, we denote $\mathbb{A}_{\lambda_i}$ and $\mathbb{A}_{\pi_j}$ as null vectors of $\mathbb{L}_{\lambda_i}$ and $\mathbb{L}_{\pi_j}$, respectively. These can be either the right or left Loewner matrices. When the samples originate from a rational matrix, $\mathbb{A}_{\lambda_i}$ and $\mathbb{A}_{\pi_j}$ relate to the null vector of $\mathbb{L}_2$, $\mathbb{A}_2$ as

$$\mathbb{A}_{\lambda_i} = \xi\begin{bmatrix}\boldsymbol{\alpha}_{i1} & \boldsymbol{\alpha}_{i2} & \dots & \boldsymbol{\alpha}_{i\overline{m}}\end{bmatrix}, \quad \mathbb{A}_{\pi_j} = \gamma\begin{bmatrix}\boldsymbol{\alpha}_{1j} & \boldsymbol{\alpha}_{2j} & \dots & \boldsymbol{\alpha}_{\overline{n}j}\end{bmatrix} \tag{C.5}$$

for some constraints ξ and γ. This is analogous to the SISO case given in Main Lemma 2.1 from [4]. Equation (C.5) implies that Theorem 4.7 holds. ■

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